Exercises for Index theory I

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In the lecture the Chern classes of a complex vector bundle $V \to M$ have been defined. These are elements $c_k(V) \in H^{2k}(M)$, a priori with complex coefficients. Recall the definition. Pick a connection ∇ on V and let $\Omega \in \mathcal{A}^2(M, \mathrm{End}(V))$ be its curvature. Its kth exterior power is an element $\Omega^{\wedge k} \in \mathcal{A}^{2k}(M, \mathrm{End}(\Lambda^k V))$. The Chern class is represented by the closed (!) form

$$c_k(V) = (\frac{-1}{2\pi i})^k \operatorname{Tr}(\Omega^{\wedge k}).$$

The Chern classes have the following properties.

1. If $f : N \to M$ is a smooth map, then $f^* c_i(V) = c_i(f^* V)$.
2. $c_0(V) = 1$ and $c_i(V) = 0$ if $i > \mathrm{rank}(V)$.
3. For complex line bundles L_0, L_1, we have $c_1(L_i) = e(L_i)$ and $c_1(L_0 \otimes L_1) = c_1(L_0) + c_1(L_1)$.
4. For two vector bundles V and W, we have $c_k(V \oplus W) = \sum_{i+j=k} c_i(V) c_j(W)$.

The goal of this exercise sheet is to prove the following result.

Theorem 1. *Let $V \to M$ be a complex vector bundle of rank n on a closed oriented manifold. Let $p \in \mathbb{Z}[x_1, \ldots, x_m]$ be a polynomial with integral coefficients. Then*

$$\int_M p(c_1(V), \ldots, c_n(V)) \in \mathbb{Z}.$$

This is a highly remarkable result, because the Chern classes were defined using decidedly nondiscrete data. A conceptual explanation can be given by algebraic topology, where the Chern classes in $H^*_{sing}(X; \mathbb{Z})$ are constructed for any bundle over any space X. One can prove that for smooth bundles over manifold, the image of the topological Chern classes under the de Rham homomorphism coicides with the Chern classes using the Chern Weil construction.

The main steps of the proof of Theorem 1 are outlined in the following exercises.

Exercise 1. (Some reductions) Without loss of generality, M is connected and p is a monomial. The proof of the Theorem will be by induction on the rank of V. Prove the induction beginning, i.e. assume that V is a line bundle. For this, make use of the fact that $c_1(V) = e(V)$ for line bundles and of the Poincaré-Hopf theorem.

Exercise 2. Let $f : N^{m+d} \to M^m$ be a smooth map between closed oriented manifolds, M connected, and let $z \in M$ be a regular value. Consider the map $f_! : H^{*+d}(N) \to H^*(M)$ constructed in Exercise 3 on Sheet 10. Let $\omega \in H^d(N)$. Prove that $\int_{f^{-1}(z)} \omega = f_!(\omega) \in \mathbb{R} = H^0(M)$.
Hint: use that $\int_N \eta = \int_M f_!(\eta)$ which is easily shown, use the formulae proven in Exercise 3 on Sheet 10 and use the Poincaré duals of $f^{-1}(z) \subset N$ and $\{z\} \subset M$. The overall argument for Theorem 1 is stable against sign mistakes, so do not worry about signs.

Exercise 3. Let $\mathrm{Fr}(V) \to M$ be the frame bundle of V; this is a $\mathrm{GL}_n(\mathbb{C})$-principal bundle. The group $\mathrm{GL}_n(\mathbb{C})$ acts (transitively) on $\mathbb{CP}^{n-1}$. Form the projective bundle

$$\pi : \mathbb{P}(V) := \mathrm{Fr}(V) \times_{\mathrm{GL}_n(\mathbb{C})} \mathbb{CP}^{n-1} \to M.$$

Prove that there is a line bundle $L_V \to \mathbb{P}(V)$ such that $L_V \subset \pi^* V$. Hint: consider the product action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathbb{CP}^{n-1} \times \mathbb{C}^n$ and observe that it preserves the tautological bundle $L \to \mathbb{CP}^{n-1}$.

Exercise 4. Prove that $\pi_!(c_1(L_V)^{n-1}) = 1 \in H^0(M)$ and more generally that

$$\pi_!(c_1(L_V)^{n-1}\pi^*\eta) = \eta$$

holds for all $\eta \in H^*(M)$. Hint: use the previous exercises and the computations of the Euler class on projective spaces.

The previous exercises together yield the important *splitting principle*, which we won't need on this sheet, though.

Theorem 2. *(The splitting principle) Let $V \to M$ be a complex vector bundle over a compact oriented base manifold. Then there exists another closed oriented manifold N, a smooth map $f : N \to M$ such that $f^* : H^*(M) \to H^*(N)$ is injective and such that f^*V splits as a sum of a line bundle L and another bundle W. By iteration, we can achieve that f^*V splits as a sum of line bundles.*

Exercise 5. Complete the proof of Theorem 1. Hint: write

$$\int_M p(c_1(V), \ldots, c_n(V)) = \int_{\mathbb{P}(V)} c_1(L_V)^{n-1} p(c_1(\pi^*V), \ldots, c_1(\pi^*V)).$$

Use the fact that $\pi^*(V) = L \oplus W$ for another line bundle and use the product formula for the Chern classes. Then use $c_1(L) = e(L)$ and the fact that the Euler class is the Poincaré dual of the zero set of a generic section. This yields the inductive step.