CONCISE ENCYCLOPEDIA OF SUPERSYMMETRY

And noncommutative structures in mathematics and physics

Edited by

Steven Duplij
Kharkov State University

Warren Siegel
State University of NY-SUNY

and

Jonathan Bagger
Johns Hopkins University

Springer

Library of Congress Cataloging-in-Publications Data

ISBN-10 1-4020-1338-8 (HB)
ISBN-13 978-1-4020-1338-6 (HB)

Published by Springer,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands.

www.springeronline.com

Printed on acid-free paper

Reprinted with corrections, 2005

Printed in the Netherlands.

Acknowledgements

It is with great pleasure that I express my deep thanks to: high-level programming language expert Vladimir Kalashnikov (Forth, Perl, C) for his extensive help; to programmers Melchior Franz (LaTeX); Robert Schlicht (WinEdit), George Shepelev (Assembler), Eugene Slavutsky (Java), for their kindness in programming, especially for this volume; to Dmitry Kadnikov and Sergey Tolchenov for valuable help and information; and to Tatyana Kudryashova for checking my English. The indispensible and timely support of my colleagues and friends is very much appreciated. Also many thanks to my wife Diana for her selflessness in patiently putting up with my everyday/night work on SUSY during the last two years.

I am sincerely grateful to the Alexander von Humboldt Foundation for both financial support and for access to modern computer equipment, which was of considerable assistance in carrying out this Project.

Steven Duplij
Editor/compiler

SUSY Story

A History of Supersymmetry narrated by its founders and precursors

FROM SUPERALGEBRA TO SUPERPHYSICS

The Story of Irreducible Representations

I graduated from the Physical Department of Moscow State University obtaining the diploma with honors in 1968. In the course of five and a half years of study there, I learned how to take the traces of the product of **gamma matrices**, learned something about *Clebsch–Gordon coefficients*, and was fascinated by the beauty of the Noether theorem. I completed my senior project under Yu.M. Shirokov, who worked at the Mathematics Institute of the Academy of Sciences. However, the Graduate School of this Institute turned me down, and the advisor of my senior project 'sold me' to the Theory Department of the Lebedev Physics Institute (FIAN). Thus, it was at the right time, in the spring of 1968, that I found myself in the right place: working with Yuri Abramovich Golfand.

His moderate height, rapid gait and a charming well-wishing smile were to level down the differences between our ages and positions. He showed me several commutation and **anticommutation relations** between the operators of momentum, angular momentum and some spinors and explained to me that their consistency was verified by **Jacobi identities**. These **commutation relations** provided the basis for the sciences which in six years was called the ***SuperSymmetry***.

At the first stage I had to establish whether the proposed algebra was unique or whether there were some alternatives. In order to do this, it was necessary to solve a system of equations for the algebra *structure constants*, which follow from the **Jacobi identities**. I restricted myself to the set of four complex spinor charges and found that there were four versions of such algebras: two of them are now known as N = 1 and N = 2 superalgebras, in two others the momentum did not commute with spinor charges (à la de Sitter algebra). We chose the simplest (N = 1) supersymmetric algebra for further investigations.

Twice a week, before seminars of the Theoretical Department, I showed the result of my calculations to Golfand, and we discussed them. At that time I used text-books on different aspects of applying group theory to physics, but I did not read any new publications. I don't know whether Golfand knew about Felix Berezin's papers or any other publications on the subject. Maybe he believed that he had already informed me about all that was necessary for our work. By all his visual appearance Golfand gave me, and probably not only me, an idea that, contrary to others, we worked on a very serious and important problem. He liked jokes and banters very much, in particular about gaps in my education.

The main problem to be solved was to relate the constructed algebra to *quantum field theory*. Nobody knew whether such a relation existed, and if yes, whether its representations were finite dimensional. I use here the word "algebra" instead of "group" for the following reason. We introduced the notion of supercoordinates and considered the group with Grassmann parameters and established the supercoordinate transformations under spinor translations. But we didn't guess to expand the **superfield** with respect to the *Grassmann variables* and to establish the relation between the **superfield** and a set of the usual fields forming the **supermultiplet**. The way to success, as it seems now, was not so elegant and, therefore, more laborious. I began to seek the representations of the algebra "at the mass shell" (without using the **auxiliary fields**).

As long as we had to do with the algebra, the main problem was to properly handle the gamma-matrices and not to forget to change the signs in certain places of the **Jacobi identities** while dealing with anticommutators. When we turned to *field theory*, we found it very difficult to get accustomed to the fact that both bosons and fermions are in the same **multiplet** and even more difficult to imagine how they transform into each other under spinor translation. There was no way to get an answer from a crib as I used to do at my exams on the Marx–Lenin philosophy. Golfand encouraged me: "A cat may look at a king." The only thing that was clear from the very beginning was that all particles in the **multiplet** have equal masses, but this fact didn't give optimism.

At last, in 1969 the *superspin* operator and two **irreducible representations** of the algebra (*the chiral multiplet* of spins zero and one half and *the vector multiplet* of spins zero and one half and one) were constructed and the following general properties of **irreducible representations** were established. First, the maximum spin in any **irreducible representation** differs from the minimum spin by at most one. The expansion turned out to have a cut-off and therefore the **irreducible representations** were finite! The second property discovered was that the numbers of bosonic and fermionic degrees of freedom in every **multiplet** were the same, and, as a consequence, the total vacuum energy of bosons and fermions was equal to zero.

I very much wanted to publish the obtained results immediately, in 1969, but Golfand believed that they would attract no attention. The psychological barrier associated with the fermi-bose mixture was broken for us exclusively. We felt sure that superinteraction will be found. A single step **from superalgebra to superphysics** was done.

So I present below a brief version of FIAN preprint no. 41 (1971): "Irreducible Representations Of The Extension Of The Algebra Of Generators Of The Poincaré Group By Bispinor Generators".

In [1] a special extension of the algebra $\mathcal{P}$ of the generators of the Poincaré group was considered. The extension was performed by introducing the generators of the spinorial translations W_α and $\bar{W}_\beta$,

$$\bar{W} = W^+\gamma_0,\ [W, \bar{W}]_+ = \gamma_\mu^+ P_\mu,\ \ [W, W]_+ = 0,\ \ [P_\mu, W]_- = 0, \tag{1}$$

where $\gamma_\mu^\pm = s^\pm\gamma_\mu$, $s^\pm = \frac{1}{2}(1 \pm \gamma_5), \gamma_5^2 = 1$.

In the same work a realization of this algebra was constructed in which a Hamiltonian describes interactions of quantum fields. This example shows that the algebra (1) imposes *rigid* constraints on the form of the quantum field interaction. In constructing this example we used two linear **irreducible representations** of the algebra (1). Their definition was not given in [1]. So now I will present the definition of these representations of the algebra (1), and will also build other representations. The properties of these representations are then investigated. Of physical interest are the representations of (1) that can be reduced to (several) representations of $\mathcal{P}$ characterized by mass and spin. Therefore, the basis vector of the space in which we build representations of (1) can be written as

$$|\kappa, p_\lambda, j, m, \chi\rangle, \tag{2}$$

where κ is the mass, p_λ is the spatial momentum, j stands for the spin, m is its projection on the z^- axis, and, finally, χ is the number of the **irreducible representation** of $\mathcal{P}$. In the space with basis vectors (2) there are subspaces invariant under the action of the operators from the algebra (1). According to Schur's lemma [2], in order to find invariant subspaces it is necessary to find invariant operators which, by definition, commute with all operations of the algebra (1). It is easy to observe that the operator P_μ^2 does have this property. Therefore, the space with basis vectors corresponding to particles of one and the same mass κ will be an invariant subspace. The spins of the vectors of this invariant subspace cannot be all equal, since the square of the spin operator is not an invariant operator of the algebra (1), $[\boldsymbol{\Gamma}_\mu^2, W]_- \neq 0, \boldsymbol{\Gamma}_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}M_{\nu\lambda}P_\sigma$. Instead, the invariant operator of algebra (1) is D_μ^2, where

$$D_\mu = \boldsymbol{\Gamma}_\mu + \frac{1}{2}\left(\bar{W}\gamma_\mu W - \frac{P_\mu P_\nu}{P_\sigma^2}\bar{W}\gamma_\nu W\right), \quad P_\sigma^2 = \kappa^2 > 0.$$

Besides the fact that the masses of all states are equal, one can assert that the difference between the maximal and minimal spins in the **irreducible representation** of (1) does not exceed 1. Otherwise, by consecutively applying the operators from the algebra (1) to the state vectors with spin j_1 one could obtain a vector with a nonvanishing projection on the state vectors with spin j_2, $|j_2 - j_1| > 1$. To see that this is impossible we note that in the most general case the consecutive action of the operators from (1) can be represented as a polynomial in these operators. The product of any number of operators $M_{\mu\nu}$ and P_λ does not change the spin of the state. The number of operators W (and $\bar{W}$) in each term can be equal to one or two, since if it exceeds 2 then the product vanishes because of the **anticommutation relation** in (1). By the same reason the product $W_\alpha W_\beta \neq 0$ if $\alpha \neq \beta$. Such an operator also does not change the spin of the state. Thus, an invariant subspace can contain only the spins j, $j + \frac{1}{2}$, $j + 1$.

We will be interested in the matrix elements of the operators W_α and $\bar{W}_\beta$, which, due to (1), are diagonal with respect to κ and p_λ. One can readily convince oneself that the operators $s^- W$ and $\bar{W}s^+$ satisfy the trivial **commutation relations**. Therefore we will limit ourselves to investigating representations in which $s^- W = \bar{W}s^+ = 0$. Without loss of generality, let us choose the representation of γ matrices with diagonal γ_5. Then the operators W_α and $\bar{W}_\beta$ become two-component. Moreover, knowing the transformation law of **spinors** under the *Lorentz transformations*, and assuming that $\kappa > 0$, let us pass to the reference frame with $p_\lambda = p'_\lambda = 0$. In this reference frame one could apply the Wigner–Eckart theorem [4], according to which

$$\langle \kappa, 0, j, m, \chi | s^+ W_\alpha | \kappa, 0, j', m', \chi' \rangle = \begin{pmatrix} j & \frac{1}{2} & j' \\ m & \alpha & -m' \end{pmatrix} (-1)^{j'-m'} \times \sqrt{(2j+1)(2j'+1)} \langle j\chi | f | j'\chi' \rangle,$$

$$\langle \kappa, 0, j, m, \chi | \bar{W} s^-_\beta | \kappa, 0, j', m', \chi' \rangle = \begin{pmatrix} j' & \frac{1}{2} & j \\ m' & \beta & -m \end{pmatrix} (-1)^{j-m} \times \sqrt{(2j+1)(2j'+1)} \langle j\chi | f^+ | j'\chi' \rangle, \tag{3}$$

where $\begin{pmatrix} j & \frac{1}{2} & j' \\ m & \alpha & -m' \end{pmatrix}$ are the Wigner symbols, $|j - j'| = \frac{1}{2}$, and $\langle j\chi|f|j'\chi'\rangle$ are the reduced matrix elements. The representation (3) ensures the correct **commutation relation** with the momentum operator and the operator of spatial rotation. In order to satisfy other **commutation relations** of (1), we substitute (3) in (1), exploit the formulas for summation of the $3j$ symbols in the spin projections [4], and obtain, after performing the summation,

$$\begin{aligned} &\sum_{JMj'\chi'} (-1)^{2j''+J+M}(2J+1) \times \begin{Bmatrix} j & j' & J \\ \frac{1}{2} & \frac{1}{2} & j' \end{Bmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & J \\ -\beta & \alpha & M \end{pmatrix} \begin{pmatrix} j & j'' & J \\ m & -m'' & -M \end{pmatrix} (-1)^{m+\alpha-j'} \times \\ &\sqrt{(2j+1)(2j''+1)(2j'+1)^2} (\langle j\chi|f|j'\chi'\rangle \langle j'\chi'|f^+|j''\chi''\rangle + (-1)^{J-j+j''} \langle j\chi|f^+|j'\chi'\rangle \langle j'\chi'|f|j''\chi''\rangle) \\ &= \kappa \delta_{jj''} \delta_{mm''} \delta_{\chi\chi''}. \end{aligned} \tag{4}$$

An analogous formula is obtained upon substitution of (3) in the **anticommutation relation** $[W, W]_+ = 0$. Next we use the values of the $6j$ symbols [4]. Then (4) takes the form

$$\sum_{j'\chi'} \left(\frac{2j'+1}{2}\right) (\langle j\chi|f|j'\chi'\rangle \langle j'\chi'|f^+|j''\chi''\rangle + \langle j\chi|f^+|j'\chi'\rangle \langle j'\chi'|f|j''\chi''\rangle) = \kappa \delta_{jj''} \delta_{\chi\chi''}; \tag{5a}$$

$$\sum_{j'\chi'} (-1)^{j'} (\langle j\chi|f|j'\chi'\rangle \langle j'\chi'|f^+|j\chi''\rangle - \langle j\chi|f^+|j'\chi'\rangle \langle j'\chi'|f|j\chi''\rangle) = 0, \quad \text{for } j \neq 0; \tag{5b}$$

$$\sum_{j'\chi'} \langle j\chi|f|j'\chi'\rangle \langle j'\chi'|f|j''\chi''\rangle = 0, \quad \text{except for } j = j'' = 0. \tag{5c}$$

These are the equations that were our goal.

First of all, starting from (5) I will deduce constraints on the number of particles in the representation of the algebra (1). To this end we multiply (5a) by $(-1)^{2j}(2j+1)/2$ and sum over $j = j''$ and $\chi = \chi''$. After this operation the left-hand side will vanish. To see that this is the case it is sufficient to transpose two factors in the second term (which is justified since this is inside the trace) and to use the fact that $(-1)^{2j} = (-1)^{2j'+1}$ (see (3)). Then the second term will differ from the first one by sign only. The right-hand side of (5a) must also vanish, and we obtain a constraint on the number n_j of particles with spin j in the representation of (1):

$$\sum_j (-1)^{2j}(2j+1) n_j = 0. \tag{6}$$

As is well known [5], in relativistic *quantum field theory* the particle energy operator, being transformed to the normal form, contains an infinite term which is interpreted as the vacuum energy. It is also well known that the sign of this term is different for particles subject to either the Bose or *Fermi statistics*. According to (6), the representations of (1) include particles with different statistics, so that the number of boson states is always equal to that of fermion states. From this it follows that the infinite positive energy of the boson states is cancelled by the infinite negative energy of the fermion states in any representation of the algebra (1).

After these preliminary remarks, we proceed directly to solving (5). Let us try to find a representation in which only particles with two distinct spin values enter. In this case j' in (5) takes only one value, and the summation over j' is, in fact, absent. Using this fact, let us multiply (5b) by $(-1)^{j'}(2j'+1)/2$ and add the result to (5a). Then on the right-hand side one will find a nonsingular matrix acting in the space of the variable χ (at fixed j, j', and j''). On the left-hand side the matrix will be nonsingular only if $j \neq 0$. (See (5c).) Therefore the representation of the algebra (1) with two spins can contain only spin-0 and spin-1/2 states. It is easy to check that in this case the simplest solution of the system with $n_0 = 2$ ($\chi = 1, 2$) and $n_{1/2} = 1$ ($\chi = 1$) has the form

$$\langle j\chi|f|j'\chi'\rangle = \sqrt{\kappa}\left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right), \tag{7}$$

where the matrix on the right-hand side acts on the state $\left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right)$. The amplitudes a and b describe the spin-0 particles while c describes the spin-1/2 particle. In the case of the three-spin representations, j, $j+1/2$, and $j+1$, the lowest spin j can be arbitrary. The simplest solution with $n_j = 1$ ($\chi = 1$), $n_{j+1/2} = 2$ ($\chi = 1, 2$), and $n_{j+1} = 1$ ($\chi = 1$) can be written in a form analogous to (7):

$$\langle j\chi|f|j'\chi'\rangle = \sqrt{\kappa}\left(\begin{array}{c|cc|c} 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \end{array}\right). \tag{8}$$

The representations (7) and (8) are irreducible. One can readily convince oneself that this is the case, even without calculating the eigenvalues of the operator D_μ^2. It suffices to observe that there exist no representations with a smaller number of particles satisfying the necessary condition (6).

In the case of the two-spin representation the operators of the algebra will be expressible in terms of non-Hermitian free scalar fields $\varphi(x)$, $\omega(x)$, and a spinor field $\psi_1(x)$. Let us show that the operator

$$W^o = s^+ W^o = \frac{1}{i}\int \left(\varphi^*(x)\overleftrightarrow{\partial_0}\, s^+\psi_1(x) + \omega(x)\overleftrightarrow{\partial_0}\, s^+\psi_1^c(x)\right) d^3x \tag{9}$$

(where the superscript o means that the operator is bilinear in the field operators while the superscript c means the charge conjugation) satisfies[1] the **(anti)commutation relations** (1). The action of the operator W^o on field operators is a linear transformation of these fields. Schematically, one can write it as follows:

$$\varphi(x) \to \psi(x) \to \omega(x) \to 0,\ \omega^*(x) \to \psi^c(x) \to \varphi^*(x) \to 0. \tag{10}$$

Let us pass now to a generalization of the representation (8) to cover the case of quantized fields. We will limit ourselves to the option that the lowest spin is zero, while two spin 1/2 particles may be considered related by the operation of charge conjugation. Then the operators of the algebra (1) in this representation can be expressed in terms of a Hermitian scalar field $\chi(x)$, a Hermitian vector transverse field $A_\mu(x)$, and a **spinor** field $\psi_2(x)$. The operator W^o in this representation has the form

$$W^o = s^+ W^o = \frac{1}{i\sqrt{2}}\int \left(\chi(x)\overleftrightarrow{\partial_0}\, s^+\psi_2(x) + A_\mu\overleftrightarrow{\partial_0}\,\gamma_\mu^+\psi_2(x)\right) d^3x. \tag{11}$$

One can verify (11) in the same manner as (9). In this representation the action of the operator W^o on free fields can be schematically depicted as

$$\begin{tikzcd}[column sep=small, row sep=small] & \chi(x) \arrow[dr] & & \\ \psi_2^c(x) \arrow[ur] \arrow[dr] & & \psi_2(x) \arrow[r] & 0. \\ & A_\mu(x) \arrow[ur] & & \end{tikzcd}$$

[1]The operator W is defined up to a phase factor, see 3.

The limit $\mu \to 0$ can be realized by abandoning the condition of transversality of the vector field and passing to the diagonal pairing $[A_\mu(x), A_\nu(y)]_- = -\frac{1}{i}\delta_{\mu\nu}D(x-y)$. In this case the field $\psi_2(x)$ becomes two-component ($s^+\psi_2 = 0$). At $\mu = 0$ the operator W^o has the form

$$W^o = s^+ W^o = \frac{1}{i\sqrt{2}} \int \left(A_\mu(x) \overset{\leftrightarrow}{\partial_0} \gamma_\mu^+ \psi_2(x)\right) d^3x. \tag{12}$$

So we have shown that the numbers of fermion and boson states in a representation of the algebra (1) coincide. Therefore, the operator P_μ^o is automatically representable in normal form. This can also be seen from the fact that the action of the operators W^o and $\bar{W}^o$ on the vacuum always yields zero, and $P_\mu = \mathrm{Tr}(\gamma_\mu^- [W^o, \bar{W}^o]_+)$. Therefore the representation (12) also possesses this property, while the vector and spinor massless particles can only be in two states with opposite chiralities. In these representations conventional fields are united in certain **multiplets**. A question arises whether one can identify these **multiplets** with some observed particles. In answering this question the main difficulty lies in the fact that the masses of all particles in the **multiplet** are equal, while this spins are different. Thus, at present the algebra (1) and its realizations must be considered as only a model of a Hamiltonian formulation of *quantum field theory*.

I express my deep gratitude to Yu.A. Golfand for his constant attention to my work and stimulating discussions.

Bibliography

[1] Yu. A. Golfand and E. P. Likhtman, JETP Lett. **13**, 323 (1971).
[2] M. I. Petrashen and E. D. Trifonov, Application of Group Theory in Quantum Mechanics (Nauka, Moscow, 1967).
[3] E. P. Likhtman, FIAN Brief Communication in Physics no. 5, 33 (1971).
[4] A. Edmonds, Angular Momenta in Quantum Mechanics, in the collection Deformation of Atomic Nuclei (Foreign Literature Publishers, Moscow, 1958).
[5] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevsky, Relativistic Quantum Theory (Nauka, Moscow, 1968).

Evgeny Likhtman

SUPERGRAVITY BEFORE AND AFTER 1976[1]

The Story of Goldstonions

In review articles and monographs on supergravity, its discovery is usually dated as 1976. However, a number of papers on supergravity appeared before 1976, beginning from 1972 [1–5]. As has now been confirmed, those papers are directly related to the established version of supergravity [6,7] and in some sense were the starting points for later ones. In this paper, I will try to fill in the historical gap (1972–1976). Supergravity is the gauged version of *global supersymmetry*. Therefore I will begin by exposing briefly those elements of **supersymmetry** which are essential for the sequel.

Supersymmetry has been independently discovered by three groups of authors: Yu. Gol'fand and E. Lichtman [8]; D. Volkov and V. Akulov [1]; J. Wess and B. Zumino [9]. The motivations and starting points used by these three groups were quite different. In [8] the motivation was to introduce a parity violation into the *quantum field theory*. The starting point of the papers [1,2] was the question whether Goldstone particles with spin one-half might exist. The authors of [9] made the generalization of the supergroup which first appeared in the Neveu–Ramond–Schwarz dual model [10,11] to the four-dimensional world.

The approach of the papers [1,2] was the most appropriate for *gauging* the *super-Poincaré group* which was done a little later in the papers of D. Volkov and V. Soroka [3,4] (1973–1974), where the **super-Higgs effect** in supergravity was elaborated. The connection of the papers [1,2] and [3,4] is very natural, as in **gauge field** theories the transformation law for the **gauge fields** is determined by the same group structure which gives a description of the *Goldstone fields.* So I will consider, as an introduction, those features of **supersymmetry** theory that are essential for its *gauging.*

A detailed exposition of the route along which the generalization of the *Poincaré group* to the super-Poincaré group was made is contained in [2]. As this paper is not well known, I will briefly recall its most essential points. As has been mentioned above, the starting point was the question whether Goldstone particles with spin one-half might exist. At the end of the 1960s, the method of *phenomenological Lagrangians* for the description of Goldstone particles, so that it reproduced the results of PCAC (partially conserved *axial vector currents*) and the current algebra, had been invented by S. Weinberg and J. Schwinger. At the time of the XIVth Conference on high-energy physics (Vienna, 1968), the problem of the current algebra and of the *phenomenological Lagrangians* had been intensively discussed (see Weinberg's rapporteur talk [12]). Two papers in the current algebra section of the conference dealt with the generalization of the method of *phenomenological Lagrangians* to an arbitrary *internal symmetry group.* One paper was presented by B. Zumino [13] (co-authors C. Callan, S. Coleman and J. Wess), and another one by myself [14]. The main results of the papers were practically identical. The difference was that in [14] the work of E. Cartan on symmetric spaces and his method of exterior **differential forms** was intensively used. In [15], which includes [14], the methods of E. Cartan were also used for the construction of *phenomenological Lagrangians* for the spontaneously *broken symmetry groups*, containing the *Poincaré group* as a **subgroup**. The Lagrangian for the *Goldstone fermions* (9) is an example of such a construction. In both papers, as well as in E. Cartan's work (and also in the many papers that followed), the decomposition of a group G into factors,

$$G = KH, \tag{1}$$

was used, with the parameters of the **coset** K forming a homogeneous space and H being the **holonomy group** of the space.

[1] D.V. Volkov (Kharkov, KIPT & CERN), CERN-TH-7226-94 (Unpublished), April 1994, 7pp. This is part of the lecture given at the TH Division of CERN and devoted to the 125th anniversary of the birthday of Elie Cartan (hep-th/9404153). See also the article 'Volkov, Dmitrij Vasilievich' in this volume.

In the method of *phenomenological Lagrangians*, the coordinates of the **coset** K correspond to the Goldstone fields. Therefore the quantum numbers of Goldstone fields coincide with the quantum numbers of the generators of the **coset** K. This fact answers the question about the possibility of the existence of Goldstone particles with spin one-half. To ensure such a possibility, the *Poincaré group* should be generalized in such a way that the generalization contains the generators with spin one-half and with **commutation relations** corresponding to the *Fermi statistics*. From a technical point of view the problem was: what representation of the *Poincaré group* is the most appropriate for such a generalization? In solving this technical problem the following representation of the Poincaré group:

$$G_{\text{Poincare}} = \begin{pmatrix} L & iXL^{+-1} \\ 0 & L^{+-1} \end{pmatrix} = \begin{pmatrix} 1 & iX \\ 0 & L \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & L^{+-1} \end{pmatrix}, \tag{2}$$

where L, L^{+-1} and X are the 2×2 matrices $L = L_\alpha{}^\beta$, $L^{+-1} = L^{\dot\alpha}{}_{\dot\beta}$, $X = X_{\alpha\dot\beta}$, turned out to have all required properties. In the generalization to the super-Poincaré group, $K_{\text{transl.}}$ plays the main role. Let us write it as consisting of four blocks:

$$K = \left(\begin{array}{c|c} 1 & iX \\ \hline 0 & 1 \end{array}\right) \tag{3}$$

Separating the blocks as

$$K' = \left(\begin{array}{c|c|c} 1 & - & iX \\ \hline 0 & 1 & - \\ \hline 0 & 0 & 1 \end{array}\right) \tag{4}$$

one can insert into the newly formed hatched blocks *Grassmann spinors* θ_α and $\bar\theta_{\dot\alpha}$ so that K' becomes

$$K' = \left(\begin{array}{c|c|c} 1 & \theta & iX' \\ \hline 0 & 1 & \bar\theta \\ \hline 0 & 0 & 1 \end{array}\right). \tag{5}$$

The matrix K' forms a group, but only under the condition that X' is complex. To satisfy the realness condition for X' with the realness condition for X in (2) and simultaneously conserving the group properties of (5), the following representation of X', as a sum of real and imaginary parts, is appropriate:

$$iX' = iX + \frac{1}{2}\theta\bar\theta.$$

The resulting expression for the super-Poincaré group is

$$G_{\text{SUSY}} = \begin{pmatrix} 1 & \theta & iX + \frac{1}{2}\theta\bar\theta \\ 0 & 1 & \bar\theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & L^{+-1} \end{pmatrix}. \tag{6}$$

From (15) one gets a transformation law for the *superspace coordinates,*

$$X' = X + i\epsilon\bar\theta, \quad \theta' = \theta + \epsilon; \quad \bar\theta' = \bar\theta + \bar\epsilon\,, \tag{7}$$

as well as the following expressions for the *left-invariant vielbein* **one-differential forms** on the **superspace** (X, θ):

$$e^a = dX^a + i\bar\theta\gamma^a d\theta, \quad e^\alpha = d\theta. \tag{8}$$

The latter are obtained as components of $K^{-1}dK$ corresponding to the generators of K – now being the supertranslation subgroup of (6). The action for the *Goldstone fermions* is an integrated four-form pulled back onto the four-dimensional *Minkowski space* (the world space) and

$$L_{GF} = \frac{1}{24}\,\epsilon_{abcd}\, e^a e^b e^c e^d. \tag{9}$$

The expressions (7)–(9) were given in [1] without a detailed deduction, which was later produced in [2]. Now, as a first step in discussing the problem of *supersymmetry gauging*, I cite the final sentence of [1]:

> "...the gravitational interaction may be included by means of introducing the **gauge fields** for the Poincaré group. Note that if the **gauge field** for the transformation (3) [formula (7) of the present text] is also introduced, then as a result of the **Higgs effect** the massive **gauge field** with spin three-halves appears and the considered Goldstone particle with spin one-half disappears."

Now let us go to the procedure of *gauging* the supersymmetry. The **gauge fields** for the *local supersymmetry* group can be introduced in the standard way:

$$A_{\text{SUSY}}(d) = \begin{pmatrix} \omega_\alpha{}^\beta(d) & \psi_\alpha(d) & e_{\alpha\dot\beta}(d) \\ & 1 & \bar\psi_{\dot\beta}(d) \\ & & \omega^{\dot\alpha}{}_{\dot\beta}(d) \end{pmatrix} \tag{10}$$

with the standard transformation law

$$A'(d) = G^{-1}A(d)G + G^{-1}dG, \tag{11}$$

where $A(d)$ is the g_{SUSY}-algebra valued **one-differential forms**. Expression (11) may be interpreted in two ways: first as the transformation law for $A(d)$, and secondly as one in which G or some of its **coset** K is interpreted as the space of Goldstone fields. In the latter case, if the Goldstone fields transform as $G'_L = LG$ (L is the left multiplication on the group G) then (11) is invariant. Analogously, if the **coset** K contains the Goldstone fields, then $K'_L = LKH^{-1}(K, L)$,

$$A'(d) = H(K^{-1}AK + K^{-1}dK)H^{-1} + HdH', \tag{12}$$

so that the projections of $K^{-1}AK + K^{-1}dK$ on the generators of K are the covariants of the **subgroup** H. The second way of interpreting (11) is appropriate for considering the spontaneously broken supergravity; the first one is useful if pure (not broken) supergravity is considered. In our case, the forms $e_{\alpha\dot\beta}(d)\psi_\alpha(d)$ and $\bar\psi_{\dot\alpha}(d)$, as well as the curvature tensor $R^\alpha{}_\beta(d, d')$; $R^{\dot\alpha}{}_{\dot\beta}(d, d'_z)$ for the Lorentz connection $\omega(d)$, are the covariants of the *Lorentz subgroup* L. The covariant forms (10) in the presence of the Goldstone fields X^a, θ^α and $\theta^{\dot\alpha}$ may be written as

$$\tilde e(d) = e(d) + \mathcal{D}X + i[(2\psi(d) + \mathcal{D}\theta)\bar\theta\theta(2\bar\psi(d) + \mathcal{D}\bar\theta)], \tag{13a}$$
$$\tilde\psi(d) = \psi(d) + \mathcal{D}\theta, \tag{13b}$$
$$\tilde\omega(d) = \omega(d), \tag{13c}$$
$$\tilde R(d, d') = R(d, d') = d\omega(d') - d'\omega[\omega(d), \omega(d')]. \tag{13d}$$

Note that in the case of infinitesimal transformations only terms linear in θ are present in (13). One can now construct the following invariant **differential four forms** contracting the indices of the **differential one-forms** [3,4]

$$W_1 = R(d_1, d_2)e(d_3)e(d_4), \tag{14a}$$
$$W_2 = \mathcal{D}\bar\psi(d_1, d_2)e(d_3)\psi(d_4), \tag{14b}$$
$$W_3 = e(d_1)e(d_2)e(d_3)e(d_4), \tag{14c}$$
$$W_4 = \bar\psi(d_1)e(d_2)e(d_3)\psi(d_4), \tag{14d}$$

using either the spinor notation ($\alpha, \dot\alpha$- indices) or the more usual vector notation for $R(d_1, d_2)$ and $e(d)$, so that W_1, W_2, W_3 and W_4 represent the Einstein action, the Rarita–Schwinger kinetic term, the *cosmological term*, and the mass term for the *Rarita–Schwinger field*, resp. The resulting action is the sum

$$W = \underbrace{a_1 W_1 + a_2 W_2}_{\text{the pure } SUGRA} + \underbrace{a_3 W_3 + a_4 W_4}_{\substack{\text{the terms due to the spontaneous} \\ \text{breakdown of the super-Poincare group}}} \tag{15}$$

The fact that the sum $a_1 W_1 + a_2 W_2$ is the pure unbroken supergravity follows from counting the degrees of freedom of the *Rarita–Schwinger field* with the action W_2 in a gravitational background. It is easy to show that if the gravitational background satisfies the equations of motion for the Einstein action W_1, then the *Rarita–Schwinger field* has two degrees of freedom. So the Goldstone fields do not contribute to the mass shell of the gravitational field. The formulas (14) and (15) and transformation law (13) are the main results of the papers [3,4].

Now, let us turn to the works on supergravity that appeared in 1976. The first of these, [6], used the second-order formalism for the Einstein action with a rather complicated transformation law for the supergravity **gauge fields**. A more simplified form of the action and transformation law has been proposed in the paper of Deser and Zumino [7]. These authors have written in their introduction:

> "The key to our results lies in the use of the first-order formalism for gravitation, in which *vierbeins* and connection coefficients are treated independently. Minimal coupling in this sense implies the existence of torsion, or of *nonminimal* contact interactions in second-order language. The first-order formulation with torsion is closely related to the description of supergravity in **superspace** [5]."

The reference note is given according to the list of references of the present paper. In the paper referred to, E. Cartan's methods of differential geometry are generalized to the *graded superspaces* for the first time ever. It is also shown that the "*flat superspace*" has torsion, and that the **holonomy group** for the curvature in the **superspace** formulation of supergravity should be the Lorentz group. The transformation laws for the **gauge fields** $e(d)$ and $\psi(d)$ used in the paper of Deser and Zumino [7] coincide with (13a,b) but were different from (13c) for the Lorentz connection form $\omega(d)$. The further development of supergravity theory has shown that the explicit forms of the $\delta\omega$ variation does not matter. As a result of further investigations, the first-order formalism and the gauged supergroup approach to supergravity based on the transformations (13) is now accepted to be the simplest way to supergravity (see, e.g., [16]). The important steps along this line of reasoning were the rheonomy theory of supergravity as well as the above-mentioned explicit proofs of the invariance of the supergravity action by using the transformation law (13), and showing that the condition $\delta\omega(d) = 0$ (13c) considerably simplifies the proof. The advantages of the first-order formalism proved useful in many aspects of the theory.

The period of intensive development of supergravity since 1976 is well described in the review article [16] and others which followed. In the text of the review [16] there is no reference to the papers [1–5]. So the present paper may be considered as an addendum to [16] with reference to the papers which were published before 1976 and were essentially based on the first-order formalism developed, in its main features, by E. Cartan.

Bibliography*

[1] D. Volkov and V. Akulov, JETP Lett. **16** (1972) 438; Phys.Lett. **46B** (1973) 109.
[2] V. Akulov and D. Volkov, Theor. Math. Phys. **18** (1974) 28.
[3] D. Volkov and V. Soroka, JETP Lett. **18** (1973) 312.
[4] D. Volkov and V. Soroka, Theor. Math. Phys. **20** (1974) 829.
[5] V. Akulov, D. Volkov and V. Soroka, JETP Lett. **22** (1975) 187.
[6] S. Ferrara, D. Freedman and P. Van Nieuwenhuizen, Phys. Rev. **D13** (1976) 3214.
[7] S. Deser and B. Zumino, Phys. Lett. **62B** (1976) 335.
[8] Yu. Gol'fand and E. Lichtman, JETP Lett. **13** (1971) 323.
[9] J. Wess and B. Zumino, Nucl. Phys. **B70** (1974) 39.
[10] A. Neveu and J. Schwarz, Nucl. Phys. **B31** (1971) 86;
R. Ramond, Phys. Rev. **D3** (1971) 2415.
[11] J. -L. Gervais and B. Sakita, Nucl. Phys. **B34** (1971) 633.
[12] S. Weinberg, Proceedings of the XIV International Conference on High Energy Physics, Vienna, 1968.
[13] S. Coleman, J. Wess and B. Zumino, Phys. Rev. **177** (1969) 2239;
C. Callan et al. , Phys. Rev. **177** (1969) 2247.
[14] D. Volkov, Kiev Preprint ITF-**69**–75 (1969).
[15] D. Volkov, Soviet J. Part. Nucl. **4** (1973) 1.
[16] P. Van Nieuwenhuizen, Phys. Repts **68** (1981) 191.

Dmitrij Volkov
(1925–1996)

*For English translation of D.V. Volkov scientific biography and his fundamental SUSY papers of 70th–80th see D.V. Volkov Memorial Page: http://www.math.uni-mannheim.de/~duplij/volkov.

A NON-LINEAR WAY TO SUPERSYMMETRY[1]

> Geometry of space is associated with mathematical group.
>
> Felix Klein, "Erlangen Program" 1872

In 2001, the science community celebrated 30 years of **supersymmetry**. I would like to recall the first steps in this direction done by our group headed by Professor Dmitri Volkov. In 1970, the Rochester High Energy Conference was held in Kiev in the Ukraine. Professor Yuriy Golfand announced two reports for this conference, but the organizing committee gave him only time for one. He preferred to discuss the problem of the vacuum in QED, since he regarded his second topic, which concerned the problem of the *super Poincaré algebra*, as very complicated. However, the abstract of this report was published in a preliminary version of the proceedings (editor L. Enkovsky), so this was the first publication about the *super Poincaré algebra*. Unfortunately, the final version of the proceedings did not contain this abstract and, moreover, the preliminary version is lost.

The mathematical background for **supersymmetry** was also set up in 1970 by Felix Berezin (Moscow) and Gregory Kats (Kiev) who published in Mathematicheskiy Sbornik (v.83, p.343) a paper about groups with commuting and anticommuting parameters. In this way supergroups entered mathematics. Volkov had investigated the connection between spin and statistics several years ago and recovered (after M. Green, 1953) the *parastatistics* in 1959. Later he considered fermionic *Regge trajectories*. The success of the application of Goldstone's theorem to π-meson physics then asked for a generalization of this theorem to the fermionic case.

Another hint was linked with Heisenberg's incorrect idea about the neutrino as a Goldstone particle connected with a broken discrete symmetry, the P-parity. At that time I was a post-graduate student and Professor Volkov proposed this problem in 1971 for my Ph.D. thesis. He headed a research laboratory in the theoretical physics division, whose director was A. Akhieser, at the Ukrainian Physics and Technology Institute (UFTI) in Kharkov. Our group also included seven post-graduate students, V. Tkach, V. Soroka, L. Gendenshtein, A. Zheltukhin, V. Gershun, and later A. Pashnev. All of these scientists left the SUSY track. The main research direction was the application of group theory to particle physics. Elie Cartan's book "Theory of Groups and Symmetric Space," was a main textbook for us during this period. Only those who finished the study of this book were able to participate in further research activities. Volkov understood that we needed this mathematical background for our goal. He said "I am afraid to reinvent the bicycle". We knew nothing about groups with anticommuting parameters. As a first attempt, we looked for an exponential representation of the *Poincaré supergroup*, but this approach was very complicated. In the Mathematics Department of Kharkov State University we participated in a seminar on mathematical physics, organized by Professor V. Marchenko. Volkov reported at this seminar on our problem and one of the participants, Professor V. Golodetz, told us about the work of F. Berezin and G. Katz connected with this new kind of group. I found this article in Mathematicheskij Sbornik, studied it, and got from its last part the example of a matrix realization of the supergroup (2|2) using *Pauli matrices*. We then constructed matrix representations of the supergroups (3|3) and (4|4). Using a well-known representation of the *Poincaré group* involving upper triangular matrices, we constructed the *extended Poincaré supergroup*, which included translations of the spinor Grassmanian coordinates and the unitary group of

[1] Dedicated to the memory of Professor Yuriy Golfand, whose ideas of **supersymmetry** inspired the most active developments in High Energy Physics over thirty years.

internal symmetries. Moreover, we constructed a **coset** space of this supergroup, which included together with 4-dimensional *Minkowski space* also Grassmanian spinor coordinates with unitary indices. This **coset** space was called "**superspace**" by A. Salam and J. Strathdee. Using the *extended Poincaré supergroup*, we constructed a nontrivial unification of spacetime symmetries, like the Lorentz or *Poincaré group*, with internal symmetries. Thus we disproved the Coleman-Mandula "no-go" theorem.

Our next step consisted of the construction of an action integral invariant under such a group and describing *spontaneously broken supersymmetry*. At that time a general theory for spontaneously broken symmetry had been developed in the papers by Callan, Coleman, Wess, Zumino (Phys. Rev. 177, 1969) and D. Volkov (Preprint ITF-69-75, 1969). We generalized it to the case of a supergroup and constructed an action for *Goldstone fermions*. This action had nonlinearly realized supersymmetries and the form of the action was Born–Infeld-like in the absence of the **gauge field** F, so that was one of the first D-3-branes, as was noticed by R. Kallosh (1997). We finished our paper in summer of 1972 and sent it to JETP Letters and to Physics Letters B. It was a big problem for our Institute, because all papers had to be checked in Moscow during three months and only after the positive decision we were able to send our paper abroad. In Fall 1972 we visited the "International mu-e seminar" in Moscow and Volkov announced a possible universal neutrino interaction. Professor E. Fradkin invited him to present a two-hour seminar at the Theory Division of FIAN. Professor V. Ogievetskiy told us about a paper by Y. Golfand and E. Likhtman that was close to our paper. But unfortunately neither Y. Golfand nor E. Likhtman had visited Volkov's seminar at FIAN. We returned to Kharkov and read the Golfand–Likhtman paper. This paper contained an action with a linear realization of **supersymmetry**. We added a corresponding reference in a longer article on the construction of matrix supergroups and the interaction of *Goldstone fermions* with other fields, and sent it to Theoreticheskaya and Mathematicheskaya Physika in Moscow (Theor. Mat. Phys. **18** (1974) 39). The breakthrough of **supersymmetry** came with a paper by J. Wess and B. Zumino (1973), but that is another story.

Vladimir Akulov

BIRTH OF SUPERALGEBRA

In the nineteen sixties the top topic was group theory and the classification of elementary particles. Bunji Sakita, the inventor of the SU(6) theory, visited us at the University of Tokyo, and excited our interest in this theory.

The SU(6) theory combines particles with spin 0 and 1, or spin 1/2 and 3/2, in one representation. I noticed a beautiful parallelism between Bose particles and Fermi particles and wondered if they could be combined in one representation. Ignoring statistics, this is easy to do. All existing elementary particles can be expressed by the adjoint representation of the SU(9) group. Hirotaka Sugawara and I wrote a short note entitled "SU(9) Symmetry" published in 1965 [1].

Of course, I was not satisfied with this scheme since this only works for one particle states, i.e., in the case of Boltzmann statistics. I came to the thought of a boson-fermion mixture just before the short note in 1965. I looked for a real mathematical scheme and soon found that a hamiltonian of the form

$$H = m\Big(\sum b^\dagger b + \sum f^\dagger f\Big),$$

where b stands for boson annihilator and f for fermion annihilator, has the usual conserved quantities consisting of $b^\dagger b$ and $f^\dagger f$. In addition, H also commutes with spinor quantities of the form $b^\dagger f$ and $f^\dagger b$. The set of all conserved quantities of these forms closes under commutators and anticommutators. Thus I arrived at a new algebraic scheme with commutators and anticommutators. This paper was published in 'Progress of Theoretical Physics' [2].

Since its publication I received suggestions that by introducing anticommuting quantities the algebra can be reduced to an ordinary Lie algebra. In such a scheme one can construct only one particle state and no more. This is the case of Boltzmann statistics and I paid no attention to them.

In 1968 I was at the University of Chicago, generalizing the algebra to include the SU(6) and introducing an algebra which I called V(6, 21). Here V meant 'beyond Unitary'. This would now be called SU(6|21). I sent it to 'Physical Review' [3]. The referee commented that this paper was very original and should be published in one form or another even though the result was not terribly interesting, and added that I should start from a simpler example. Actually this was already done in my previous paper, so I stuck to the complicated model.

While writing this paper I did not know how to call an algebra of this type. One day I called a Professor of Mathematics, Ichiro Satake, explained my algebra and asked him if such a commutator-anticommutator algebra exists in the mathematical community. He replied that an algebra with anticommutators only was often called a Jordan algebra, but he had never seen such a mixture. He was not interested in this algebra.

About the same time I also explained this to Murray Gell-Mann, who remarked: what are all types of such an algebra? The compact *Lie groups* are limited to a few cases: orthogonal groups, unitary groups, symplectic groups and some exceptional ones. Similarly, all types of the mixture algebra could be listed up. I regarded this an interesting mathematical problem.

However, I wanted to try a more physical project, i.e., to formulate the scheme in a relativistically invariant way. I first tried to write down an example of a relativistically invariant Lagrangian that accepts the *boson-fermion symmetry*. This was not easy, and before reaching the goal I lost interest in this project. I thought that such relativistic *boson-fermion symmetry* (now called supersymmetry) could be formulated mathematically, interesting and exciting, but it would not be the fundamental symmetry of physics. If it were, the fundamental particles must consist of fermions and bosons. This contradicts the principle that the fundamental objects must be very few. Also, I think that the nonrelativistic *boson-fermion symmetry* is important for physics.

Bibliography

[1] H. Miyazawa, H. Sugawara, Prog.Theor. Phys. **33** (1965) 771.
[2] H. Miyazawa, Prog. Theor. Phys. **36** (1966) 1266.
[3] H. Miyazawa, Phys. Rev. **170** (1968) 1586.

Hironari Miyazawa

THE STORY OF SUPERALGEBRAS OF FIELD OPERATORS

The objects subsequently called superalgebras were introduced in the works [1,2]. The starting point was an attempt to answer the question of what the local interactions of quantum fields are in the Heisenberg picture the (anti)commutators of fields shifted in time are not c-numbers in the presence of interaction. In [1] the conjecture was made that in a suitable formalism of the fields, after restriction by 3-linear vertices, some averaging of local operators and in first order of expansion in Δt of the local algebra of the fields A_i, one can give the closed form

$$[A_i, A_k]_{\pm} \sim c\text{−terms} + \Delta t \cdot P_{ijk} \cdot A_k, \tag{1}$$

where $P_{ijk} \neq 0$ are certain matrix operators including the matrix of interaction parameters. In doing so the commutator operation will connect Bose (b) and Fermi (f) elements of the algebra by the rules

$$[f, f]_{+} = b, \;\; [f, b]_{-} = f, \;\; [b, b]_{-} = b, \tag{2}$$

which leads directly to the superalgebra structure. Actually, for fields (as distinct from currents) it must be the algebra of causal connection of two different timelike points and the commutator operation must be constructed in terms of antisymmetrized point by point (anti)commutators, as explained in [3,4]. Already such a scheme will not be a superalgebra, though Bose and Fermi elements enter in it. The construction of a similar model as a whole requires learning the rigid spacetime geometry and making a transition to the algebra of fields on causal graphs. The corresponding starting definitions were given in [3,4]. At present there are grounds to suspect that such a fermion-bosonic, locally Lorentz-covariant field model exists, while in it a high energy *gauge group*, boson and fermion multiplets, the dimension of spacetime and the number of fermionic generations are fixed.

In the works [1,2] attention was attracted to a mathematical problem that instantly arises when trying to realize the overall algebra of the fields in terms of relations (1) and (2), that is, the necessity of considering nondegenerated structures in which the commutator operation translates Bose and Fermi elements one into another by the rules of physical grading. Though the corresponding generalized Jacobi identities were known, at that time it seems nobody tried to study and classify such constructions as analogs of Lie algebras.

Therefore, based on the above physical considerations in [1,2] such an abstract Lie-like algebraic construction was introduced and described, and some of its structure properties were analyzed, allowing us to distinguish a class of irreducible algebras. In the final part of [1] a concrete example that subsequently turned out to be a minimal *simple Lie superalgebra* was constructed.

At that time I communicated with Victor Katz, who called me (somewhere in 1971) with a question about an algebra with a given number of generators. I found such a construction among my old notes, which I had stopped working out. That attracted his attention to this subject, which was subsequently developed by him in detail [5].

The first ground-laying thoughts [1] arose in February 1965, when I wondered whether the right-hand side of the field commutators should not be c-numbers, but some third field. At that time there were many active discussions on the axiomatic construction of *quantum field theory*, A. S. Wightman and the theory of R. Haag about the non-existence of an interaction representation (reviewed in lectures by I. T. Todorov in Dubna, October 1964 [6]). Here, the principal fact for me was a conclusion about the connection between interaction of the fields and their simultaneous (anti)commutators. I started to think about the possibility of unifying the Heisenberg equations of motion and commutator initial conditions on space-like hypersurfaces into something common. At any small time shift this really is possible, and when phrased in spacetime arguments and restricted to linear contributions of 3-linear interactions, a nonsingular unifying algebra of fermionic and bosonic elements will emerge, with the commutators as generators.

While reading the review by I. V. Polubarinov (Dubna, November 1965 [7]) I learned about the formulation of quantum electrodynamics in terms of tension [8], where the simultaneous commutation relations contain nontrivial boson-fermion one of type, $[f, b]_{-} = f$. It make me certain of the usefulness of a detailed consideration of this nonsingular fermion-boson commutator algebra, with possible inclusion of fermion-fermion brackets.

In June 1965 as along this cause I phoned E. B. Dynkin on this matter, who, apologizing since he had to leave, sent me to F. A. Berezin. At that time I did not know about the work by F. A. Berezin, and therefore I decided not to address him. During the winter of 1965–1966 there were contacts with V. I. Shelest (Vice Director of the Kiev Institute of Theoretical Physics), who invited me to give a talk at the International School in Yalta in April 1966. During the School meeting there were discussions with M. A. Naimark, I. M. Gelfand and I. T. Todorov (the last was a chair at my report). On initiative of V. I. Shelest my report was published in the Proceedings of the Yalta School in 1967, and a Kiev ITF preprint was issued [2].

After that, during 1968–1969 I considered special cases in the classification of simple superalgebras, which I then called K-algebras. However, the profound professional activity of V. Katz in this direction made publication of the models superfluous. In general, at first I did not know how to call my operator field algebras, and I wanted to name them 'field algebras', but was reluctant. Later I learned that in 1967, Lee, Weinberg, Zumino called similar algebras (but involving only bosonic fields and only commutators) 'gauge field algebras' [9].

Also, somewhere in Spring 1970 I gave a seminar at the Lebedev Physical Institute, Moscow, with a detail explanation of my ideas. Later I learned that Yu. Golfand and E. Likhtman were there, but at that time I did not know them personally.

As is well-known, during 1971–1972 supersymmetry started to evolve stormily. Nevertheless, e.g. Bryce De Witt wrote [10]: " The author is indebted to G. A. Vilkovistsky for the following bit of historical information: the concept of super Lie algebras appears to have been introduced for the first time by G. L. Stawraki (see ref. [1,2] here). Stawraki called them 'K-algebras' after the first letter of his wife's name".

Bibliography

[1] G. L. Stavraki, Some nonlocal model of field selfinteractions and the algebra of field operators, in Proceedings of International School of Theoretical Physics, Yalta, April-May, 1966, "High Energy Physics and Theory of Elementary Particles", Naukova Dumka, Kiev, 1967, p. 296.
[2] G. L. Stavraki, On a generalization of Lie algebra, preprint ITF-67–21, Kiev, 1967.
[3] G. L. Stavraki, Description of quantum fields as carriers of space-time causal structures and internal symmetries, Univ. Texas preprint, Austin, December 1989.
[4] G. L. Stavraki, Theor. Math. Phys. **84** (1990) 339.
[5] V. G. Kac, Lie superalgebras, Adv. Math. **26** (1977) 8–96.
[6] I. T. Todorov, in Proceedings of the International School of Theoretical Physics, JINR, Dubna, vol. **1**, 1964, p. 5.
[7] I. V. Polubarinov, Equations of quantum electrodynamics, preprint n. P2421, Dubna, 1965.
[8] S. L. Hammer, R. H. Good, Ann. Phys. **12** (1961) 463.
[9] T. D. Lee, S. Weinberg, B. Zumino, Phys. Rev. Lett., **18** (1967) 1029.
[10] B. De Witt, Supermanifolds, Cambridge Univ. Press, 1987, p. 172.

Georgij Stavraki

THE STORY OF SUPERALGEBRAS

I began working on Lie superalgebras in 1969 after I met Georgii Stavraki. Somehow a group of physicists learned about my thesis published in [1] where I introduced Kac-Moody Lie algebras (under the name contragredient Lie algebras) as a part of a classification of graded infinite-dimensional Lie algebras. I guess, the word ‘‘graded’’ confused physicists, they thought that the thesis was about what was later called superalgebras. They invited me to give a talk at some Moscow institution. I gave the talk (some time in 1969), and after the talk several people explained to me what they meant by a ‘‘*graded Lie algebra*’’, and Stavraki showed me his example published in [2]. This was a *simple Lie superalgebra* $sl(2|1)$ written down in a basis in terms of explicit commutators.

So I think Stavraki is by far the first physicist who published a field model using Lie superalgebras in 1967 (in the Proceedings of 1966 Yalta International School of Theoretical Physics) [2]. After that we have had several discussions, but neither of us could understand what the other was saying. At any rate, in 1971 I published my first results on the classification of Lie superalgebras in [3]. Incidentally, I went to this conference only because it took place in my native town Kishinev (Moldova).

After reading the paper [4] by F. A. Berezin and G. I. Kac it became clear to me that it was worth publishing my results; before that I thought this was just a curiosity. I talked to Berezin some time in the early seventies and showed him a photocopy of my 2-page paper [3]. He told me that I should publish immediately all the results that I had in a good journal. I remember him saying ‘‘Vasha zametka goditsia tol'ko na samokrutku’’ (your note is good only for a self-made cigarette), probably referring to its inaccessibility, and ‘‘Zemlia gorit pod nogami’’ (land is burning under your feet), probably referring to the explosion of interest to supersymmetry at the time. This is how I learned that supersymmetry had become a popular subject. I told him that I would publish only after I had a complete solution. After talking to Berezin, I called Stavraki and told him about our conversation, suggesting that he should publish whatever he had. Stavraki replied that he'll do it only after he had a complete theory. Unfortunately, as far as I know he never published anything on supersymmetry except for the 1967 paper.

As for me, I finally succeeded in getting a complete classification. This happened at the beginning of 1975. The main idea was incredibly simple and beautiful: to use the canonical filtration introduced already by Cartan in his study of infinite-dimensional Lie algebras of vector fields and developed by Guillemin and Sternberg, but never used in finite-dimensional theory. It sounds kind of trivial to apply in the finite-dimensional situation the infinite-dimensional techniques, but the idea turned out to be extremely powerful along with the idea that a subspace of a superspace can be mixed, not only purely even or purely odd, which is, of course, another triviality, but a very important one.

Thus I published an announcement [5], and in September of 1975 I submitted the full version to Uspekhi Matematicheskih Nauk (Russian Math. Surveys). After physicists introduced the catchy word ‘‘supersymmetry’’, it was only natural to use in my papers the words ‘‘superalgebras’’, ‘‘supertrace’’, etc. I should mention that, as always, I got a lot of help from my former advisor Ernest Borisovich Vinberg in this very difficult work. Incidentally, at the time I also tried to classify infinite-dimensional superalgebras, but succeeded only 23 years later (see [6]). Unfortunately the publication of the full version in Uspekhi was being delayed, and after I emigrated to the US it became, of course, impossible to have it published in a Soviet journal. So I gave the paper for publication to my newly acquired colleague at MIT, G.-C. Rota. The paper was finally published in Advances in Math. in 1977 [7]. Rota insisted that no ‘‘political statement’’ should be made, so my explanation of the reason of withdrawal from Uspekhi was deleted. At any rate, the paper circulated in the West since early 1976 due to the efforts of S. Sternberg. The paper was smuggled from

Russia and given to Sternberg by a visitor of Moscow University from Harvard in the fall of 1975, and Sternberg arranged its translation and distribution.

Let me conclude with a couple of entertaining stories. I am often asked why I denoted the "strange" series of Lie superalgebras by P and Q. I lectured on the classification at Moscow University some time in 1975. Berezin, Kirillov and Palamodov, among a few, were present. When I arrived at the "strange" series, the question arose how to denote it. I said that I wanted to denote it in honour of someone present. Berezin was ruled out since there was already the series B, Kirillov was ruled out since there was the series of contact algebras denoted by K, so I denoted it by P in honour of Palamodov. After that it was only natural to denote the remaining series by Q (though, you can imagine that there was nobody in the room with the name beginning with Q, as Q is not a letter of the Russian alphabet).

There were a few funny encounters with physicists in the early 70's. The most memorable one was with Zel'dovich. He invited me to his house and I explained him the superalgebras with great enthusiasm. After a while he asked me: what do you call an algebra? I replied: a vector space with multiplication. Then he said: but isn't it true that the vector multiplication exists only in the 3-dimensional space? Of course, I should have explained to him the thing in terms of the superbracket of operators.

One can find a brief history of my paper on the classification of finite-dimensional Lie superalgebras [7] at the end of the introduction.

Bibliography

[1] V. G. Kac, Simple irreducible graded Lie algebras of finite growth, Math USSR Izvestija **2** (1968) 1271.

[2] G. L. Stavraki, Some non-local model of field selfinteractions and the algebra of field operators, in "High Energy Physics and the Theory of Elementary particles", Naukova Dumka, Kiev, 1967, p. 296.

[3] V. G. Kac, Some algebras related to quantum field theory, in "Ith All-Union Algebra Coll.", Kishinev, 1971, p. 140.

[4] F. A. Berezin, G. I. Kac, Lie groups with commuting and anticommuting variables, Math. USSR Sbornik **11** (1970) 311.

[5] V. G. Kac, Classification of simple Lie superalgebras, Funct. Anal. Appl. **9** (1975) 263.

[6] V. G. Kac, Classification of infinite-dimensional simple linearly compact Lie superalgebras, Adv. Math. **139** (1998),1–55.

[7] V. G. Kac, Lie superalgebras, Adv. Math. **26** (1977) 8–96.

Victor Kac

THE STORY OF SUPERDETERMINANTS

In 1967–68, following the publication of the note by F. A. Berezin "Automorphisms of Grassmann algebra" [1], Grigorij Kats from Kiev (unfortunately, I did not know him) stated the conjecture that the analog of Berezin's formula for the *Grassmann algebra*,

$$J_{Berezin} = \frac{1}{\det D}, \tag{1}$$

for a tensor product of Abelian and *Grassmann algebras* becomes

$$J_{Kats} = \frac{\det\begin{pmatrix} A & B \\ C & D \end{pmatrix}}{\det^2 D}, \tag{2}$$

using the notation of my paper [3]. To be absolutely accurate, indeed F. A. Berezin gave me Kats' conjecture. At that time I was a student in the fifth grade. Approximately in May 1968, before last exams, I was "striked"and I derived the main results, guessing how to expand the automorphism into a composite of triangle transformations. I told Berezin about it, but he did not show any special interest and suggested I write out everything first. Next there were exams, admission to the graduate school, etc. Somewhere at the end of 1968, or rather in the beginning of 1969, I wrote the draft version of the work and gave it to F. A. Berezin. First he proposed that I applied the results to the analysis of the 3-dimensional *Ising model* by analogy with the 2-dimensional case (the paper which he just prepared and gave me to read before publication [3]). I messed around with it for a long time, but was not able to find something interesting for four-forms. Ultimately, I lost interest in this subject.

After graduation I was assigned to the faculty of economics of Moscow State University, due to efforts of academicians A. N. Kolmogorov and P. S. Alexandrov. This is a special story: the local Communist Party Committee of the Mathematics Department considered me unreliable and they did not want to have me stay there. But I was sure that my result confirms the conjecture of G. Kats (2). One day F. A. Berezin called me and said that the work had to be published urgently, that science does not stop at one result, that people derive results, and they should cite mine. In general, he was right.

I promptly wrote a paper and submitted it in 1973 to the journal "Mathematical Notes", since the paper continued Berezin's article published in the same journal in 1967 [1]. At first it was rejected: a referee wrote that nobody needs it (and me — implicitly), and moreover, he does not like the logic of the proof of Lemma 2. Berezin insisted, and I answered the referee rather bluntly that we may not judge what is and what is not necessary for science, and explained that the logic of the proof is correct, following an example of analogous proving "at secondary school level". The paper was accepted, and it appeared in 1974 [2]. Regrettable, the editor's interference created ground for some fantasies. That is, at the end of the paper I wrote that "F. A. Berezin gave the author the conjecture by G. I. Kats and helped with advice. The author of the paper thought that indeed this conjecture was proven by him. However D. Leites pointed out that the result obtained does not coincide with the conjecture of G. Kats." I am not aware of the thoughts of the editor (a begining author had better not discuss with him), but the above acknowledgment appeared in the journal as "The author is thankful to F. A. Berezin for posing the task and for help, as well as to D. Leites for useful discussions".

Thus, the formula obtained by me in 1968 is

$$J = \frac{\det(A - BD^{-1}C)}{\det D}. \tag{3}$$

One may notice here that formula (2) is not thoroughly well-defined because it contains anticommuting (odd) parameters (in B and C), and the ordinary determinant cannot be used until an ordering rule is given for them, while formula (3) contains determinants of even (commuting) variables only, for which ordering is not needed, and so it is well-defined.

At the time I had a meeting with academician S. P. Novikov in Chernogolovka (Landau Institute of Theoretical Physics, Moscow region), who told me that J. Schwinger had obtained all of my results long ago and promised to give me all references next day. Unfortunately, when I called him at the agreed time, he said that he had teased me and that Schwinger had proven similar but somewhat other things, and he had no references at hand. I did not put due attention to this event. Later I learned that Schwinger tried to consider commuting and *anticommuting variables* on a par in his physical constructions, as well as groups unifying such parameters, and he already used an operator analog of formula (1) in 1955 in his Les Hauches lectures [4] (see also *anticommuting variables*, prehistory in physics).

This has been the overall setting. Fairly speaking, it was not very desirable to recall all that, moreover since F. A. Berezin tragically perished. He did a whole lot of good for me. Also, I myself feel uncomfortable towards G. Kats for the fact that in due time I did not insist on my own formulation.

Bibliography

[1] F. A. Berezin, Math. Zametki, **1** (1967) 3 (Math. Notes **1** (1967) 180).
[2] V. F. Pakhomov, Math. Zametki **16** (1974) 65 (Math. Notes **16** (1974) 624).
[3] F. A. Berezin, Uspehi Mat. Nauk **24** (1969) 3.
[4] J. Schwinger, Lectures at the Summer School of Theoretical Physics, Les Houches, France, 1955; J. Schwinger, Quantum Kinematic and Dynamics, Benjamin, New York, 1970.

Valerij Pakhomov

THE STORY OF CENTRAL CHARGES

In 1974 I got an invitation from Julius Wess to visit in summer the University of Karlsruhe for several weeks. I was already aware of the very important work done by Julus Wess and Bruno Zumino on supersymmetry. After my arrival I got immediately very interested in these fascinating developments. To outline my modest contribution to this theory I have to quote a few formulas; at that time they looked as follows:

$$\left\{ Q_A^{(L)}, Q_{\dot{B}}^{(M)+} \right\} = c\, \delta^{LM}\, P_{A\dot{B}}, \quad c > 0,$$

$$\left\{ Q_A^{(L)}, Q_B^{(M)} \right\} = 0, \tag{1}$$

where $A = 1, 2$; $\dot{B} = 1, 2$; $L, M = 1, 2, \ldots, N$, with N an integer number; $Q_A^{(L)}$ and $Q_A^{(L)+}$ denote a *spinorial charge* and its resp. hermitean conjugate in *quantum field theory*; and

$$P_{A\dot{B}} \equiv \sigma^{\mu}_{A\dot{B}}\, P_{\mu}\,, \tag{3}$$

where (P_0, P_1, P_2, P_3) form the energy-momentum operators. Finally, the curly brackets stand for an anticommutator. After studying the current results it occurred to me that the r.h.s. of (1) does not need to vanish. I can remember that one evening I phoned Julus Wess and presented him my problems. Julius reacted immediately by encouraging me to continue to develop my idea. I answered him that I would like very much to do so, but that I am not yet too well acquainted with the whole structure of the spinon calculus. So he suggested to me that he will ask Dr. Martin Sohnius to assist me in my work; Dr. Sohnius was very good as far as technical problems of the spinon theory were concerned. Hence started my collaboration with Martin. We suggested first that the formula (1) can be extended to

$$\left\{ Q_A^{(L)}, Q_B^{(M)} \right\} = \epsilon_{AB}\, Z^{(LM)} + b^{(LM)}\, M_{AB}, \tag{2}$$

where

$$Z^{(LM)} = -Z^{(ML)} = \sum_{l=1}^{n} a_l^{(L,M)}\, B_l \tag{3}$$

and B_l denotes the l–th scalar charge (n finite), which appears in *quantum field theory* as *internal symmetry generator*,

$$M_{AB} = -(\sigma^{\mu\nu})_{AB}\, S_{\mu\nu}$$

where $S_{\mu\nu}$, $(\mu, \nu) = 0, 1, 2, 3$, are the generators of the Lorentz group. Finally, $b^{(LM)}$ and $a_l^{(LM)}$ are certain numerical coefficients. However we soon realized that $Q_A^{(L)}$ commutes with the translational group and that M_{AB} does not. So relation (2) reduced to

$$\left\{ Q_A^{(L)}, Q_B^{(M)} \right\} = \epsilon_{AB}\, Z^{(LM)},$$

with $Z^{(LM)}$ given by (3). We wrote a preprint in July 1974 entitled “On the Algebra of Super-Symmetry Transformations”. At that time we did not yet know that $Z^{(LM)}$ are **central charges** of the whole supersymmetric algebra.

Then I left Karlsruhe for CERN in Geneva. There in CERN, I met Bruno Zumino, another great creator of the **supersymmetry** theory in *quantum field theory*. But there was also Rudolf Haag, a famous physicist who also got deeply interested in *supersymmetric theory* in *quantum field theory*. So we started our collaboration of these topics. Later Martin Sohnius joined us at CERN. The result of our work was a compete, systematic description of all symmetries and **supersymmetries** in *quantum field theory*, based on the famous "no go" theorem stated by S. Coleman and J. Mandula (Phys. Rev. **159**, 1251 (1967)) and concerning the *S-Matrix* symmetries. The title of the paper by Rudolf, Martin and me, published in Nuclear Physics **B 88**, 257 (1975), was deliberately chosen by us so that it stressed the close links to the work of Coleman and Mandula, mentioned above, namely "All Possible Generators of Super-Symmetry of the *S-Matrix*". Some people say that it is a good paper. This seems to be a proper place to close my note.

Jan Lopuszanski

SUPERSYMMETRY, CONFORMAL INVARIANCE AND INTERNAL SYMMETRIES

The story of Internal symmetry

In September 1975 I followed an invitation to spend a year at CERN. It was mainly V. Glaser with whom I shared interests and intended to work with. But I met there many other friends of the past years. In particular there was Bruno Zumino, who introduced me to the virtues of **supersymmetry**:

1) cancellation of some nasty terms in the perturbation expansion;

2) the fact that the basic relations in the Wess–Zumino model already implied positivity of energy, a property of paramount importance in QFT which previously had to be brought in by assumption;

3) the possibility of bypassing older **no-go theorems**, which forbid a nontrivial fusion of geometric and internal symmetries.

In spite of these virtues I did not jump to **supersymmetry** immediately. It was the visit of Jan Lopuszanski to CERN that confronted me with the simple question: "Why could there not be any *spinorial charges*?" Indeed, why not? Of course, you could not measure them. But they might serve as generators of an algebra of symmetry operators. Jan had come from Karlsruhe where he had written a paper together with Martin Sohnius, then a young postdoctoral fellow in the group of Julius Wess. They had extended the charge structure of the Wess–Zumino model by introducing "**central charges**". So I decided to join their efforts for a systematic study of possibilities.

The first objective was item 3) above, finding out how far **supersymmetries** modified the theorem of Coleman and Mandula about possible symmetries of an S-matrix. This was rather easy, but the result was not very exciting. I felt that, if *fermionic charges* were at the root of symmetries then they should not only generate the translations, but also the Lorentz group together with additional internal symmetries. Apparently this was not possible in this setting. It turned out, however, that if one did not focus on an *S-matrix* involving massive physical particles, then there is a very elegant and natural extension in which the *fermionic charges* generate the *conformal group* together with a restricted set of internal symmetries (essentially the $SU(n)$ groups). I want to acknowledge a very helpful discussion with Gian Fausto Dell Antonio which put us on this track. The joint paper by Lopuszanski, Sohnius and myself, entitled "All possible symmetries of the *S-matrix*" was ultimately finished after a visit to Karlsruhe and finally out to Wroclaw, where Jan, loosing patience with my tardiness, commented: "Now you sit down here and start writing".

In the sequel I tried for some time to proceed from *spinorial charges* to spinorial currents, hoping that they could generate a full supersymmetric field theoretic model. These efforts, partly together with Martin Sohnius and partly with Klaus Fredenhagen, were not rewarded by any conspicuous success. So I gradually drifted away again from **supersymmetry** though I still believe that the algebraic scheme in the paper with Lopuszanski and Sohnius could be relevant in some future context.

Rudolf Haag

SYMMETRIES WIDER THAN SUPERSYMMETRY

Towards noncommutative and nonholonomic geometry

> Dictionaries are like watches. The worst is better than none, and the best cannot be absolutely true.
>
> *Samuel Johnson*

In this foreword I will try to give a highly personal overview and sum up in a few words what is done to this day on Supersymmetry and Noncommutative Structures and what is NOT covered in the Concise Encyclopedia, nor in [44]. Following the old slogan "Let all flowers blossom", tested during Mao's culture revolution, the editors did not prevent the authors to dwell on their results (sometimes at the expense of some deeper "rival" results, Berezin's included).

My primary concern is to list main **names** rather than results: with Internet facilities the reader who knows whom to read will be able to do the rest, and it is often the best to read the classics' original papers, however incomprehensible (like Newton's papers in Latin or Lie's papers in German or some of **Berezin**'s papers) and littered with mistakes (if taken too literally) they might be. For example, it does not matter what, say, **Witten** or **Drinfeld** wrote in his latest paper: it is worth reading anyway if one is interested in the latest (and future) developments of physics and mathematics. (Still, some knowledge and books, say, authored by Hj. Tallqvist, an expert in nonholonomic mechanics, are half-sunk in Lethe and can only be dug out of ancient referee journals. Even a masterpiece [27] by F. Klein and Sommerfield on tops is not translated from German.) For the lack of space, I could not list all the names nor references I'd like to; e.g., on M**-theory** (see a review [45] and other works by **A. S. Schwarz**), D-modules, etc.

Prehistory There lived several giants some of whose ideas and constructions only recently started to contribute in earnest to modern language and Gestalten of today's physics. **Grassmann**, who defined *Grassmann algebra*, not only, see [37]. **Hertz**, known for the unit of frequency, who, in order to eliminate the notion of force, and initiated "geometrodynamics" and introduced the notion 'nonholonomic' for dynamical systems with nonintegrable constraints on velocities, see [25]. More generally, a nonholonomic manifold is a manifold endowed with a nonintegrable distribution. Since any model of *Minkowski superspace* is nonholonomic, this becomes pertinent to us, fans of super. But not only to us: examples of nonholonomic systems are ubiquitous and range from cars and falling cats to pursuing missiles to supergravity to statistical physics to economics to optimal control, etc., so possible applications (not only for grants) are promising, cf. [21]. **Riemann**, whose tensor stands in the left-hand side of Einstein's equations, the generalization of this tensor for any G-structure and further generalization to manifolds and supermanifolds with nonholonomic structures is vital for us. **Lie**, who introduced not only *Lie groups* and their "tangents", **Lie algebras**, but also a form rediscovered by Berezin and **Kirillov** and important in Kirillov's description of classical mechanics. **Weyl**, who introduced **gauge fields**, later generalized to become **Yang–Mills** fields (to the mathematician these are just vector bundles with connections). **Planck** and **Heisenberg** who enforced quantization and prompted development of the "**noncommutative geometry**".

On symmetries and unification: **Maxwell** unified electric and magnetic forces; **Poincaré** whose group is supposed to be the group of symmetries of all fundamental forces and which unites time with space; **Glashow, Salam and Weinberg** unified electromagnetic and weak forces.

Veblen's problem: classification of invariant differential operators (see **Kirillov**'s elucidations [31] and Grozman's solution of particular cases [23]). Importance of this problem dawned after Einstein demonstrated invariance of Einstein equations with respect to the group of diffeomorphisms. In *field theory*, an essential role is played by the invariance of the properties of elementary particles with respect to a transformation group. The invariance of these properties was instrumental in developing a pattern in the zoo of elementary particles. The efficiency of this approach was due to its capacity to predict properties of particles as yet undiscovered; these properties were eventually confirmed by experiments.

Weil's "proche points" = near points is a key word of the working language of modern supermanifold theory and **Grothendieck**'s revolution in **algebraic geometry**: spectra and *schemes*. For the best and briefest introduction see **Manin**'s lectures [33]. In these lectures it is explicitly stated that one-point spectra may have inner degrees of freedom, like elementary particles.

Introduction of **strings** seems to be the first, after Anaxagoras (ca 500–428 BC), idea that "atoms" are not points. On catch-26 pertaining to *string theory* and how to resolve it by means of **Kaluza–Klein**'s ideas, see [18]. (Regrettably, one seldom if ever acknowledges **Mandel**, perished in 30's, who also developed same ideas at the same time and, moreover, united them with nonholonomicity, cf. [28].)

Penrose's *twistors*: passage from the *Poincare group* to its complexified deformation, a simple group $SL(4;\mathbb{C})$, for a *superization* see **Manin**'s [36]. Observe that *supertwistor* models naturally lead to nonholonomic structures.

Supersymmetry: first calls: models by Stavraki, Golfand and Likhtman, Volkov and Akulov; superalgebras of Neveu and Schwarz, and Ramond.

History. The first human who realized that he was entering a new field — "*supermathematics*" — was, undoubtedly, **Felix Aleksandrovich Berezin**. Working with questions of second quantization he noticed that it is possible to carry out a parallel description of Bose and Fermi fields, and in mid-sixties came to the conclusion [2] that there exists a nontrivial analogue of Calculus in which the elements of a *Grassmann algebra* play the role of functions. In particular, this meant that our universe is not a space locally equivalent to the *Minkowski space*, but is a superspace. **Wess** and Zumino were the first to consciously demonstrate that **"we live on a supermanifold!"** and some amazing consequences. It was only recently that mathematicians realized that this supermanifold is neither real nor complex and introduced a rigorous notion of complex-real supermanifold, see **J. Bernstein**'s lectures in [11].

— It is universally accepted nowadays that "all" particles fall into two categories: bosons (particles with integer spin satisfying the *Bose–Einstein statistics*) and fermions (particles with half-integer spin satisfying the *Fermi–Dirac statistics*). Recent studies, e.g., of high temperature superconductivity, demonstrate that there are particles more general than Bose and Fermi particles — **anyons** or paraparticles — which obey *parastatistics* rules and whose "spin" is an arbitrary number, (see **Majid**'s elucidations). A fundamental hypothesis of theoretical physics is that all interactions are invariant with respect to the Poincaré group, the group of spacetime transformations in the relativity theory. Under this hypothesis it is impossible to unite bosons and fermions together into one **multiplet** (grouping of particles, indistinguishable in interactions of a given type): one of **no-go theorems**. (Though false as stated, as is clear from, e.g., [11,13,36], these "theorems" inspired further research.) Physicists have long desired to unify particles of different statistics into single **multiplets**, in order to decrease the number of the truly elementary "building blocks" which constitute matter. This can be done by introducing supergroups.

— In topology, *supercommutative superalgebras* have been known for a long time. However, the theory of such algebras has not solicited any special attention. It is also in topology, as well as in various deformation theories, that certain algebras (**Lie superalgebras**), whose structure resembles that of **Lie algebras**, have made their appearance.

It was also noted that there is a relationship, similar to Lie theory, between **Lie superalgebras** and **Hopf algebras**.

— Classification (ca 1975) of simple finite dimensional complex **Lie superalgebras** due to **Nahm**, Rittenberg and Scheunert, and I. Kaplansky, and skillfully rounded up by **V. Kac** together with

V. Kac's exposition of some elements of their representation theory, both strikingly similar to those of finite dimensional **Lie algebras** (and to some infinite dimensional, namely, $\mathbb{Z}$-graded and of polynomial growth) were extremely impressive and timely.

– Mathematicians also got more interested in **supermanifolds**: mathematics connected with or initiated by E. Witten's papers on **supersymmetry**; **Quillen**'s "**superconnections**" as a natural language for the *index theorem* and elaboration of these works by Alvarez–Gomé, Bismut, Getzler [6], Manin (to mention a few) elucidated even the conventional theorems.

– **Witten** and A. Raina consistently used Weil's near points or the language of families or odd parameters.

Superstructures in the "classical mathematics". The majority of the structures indicated below are not understood. By this I mean that no "practical advantage" is derived from their existence. The lucky exceptions are the index theory ([6]; though its *superization* still is an open problem and the classical invariant theory à la **Roger Howe** (see a review in [13]): in super terms we get a simpler proof of an old result or the hart of the matter becomes clearer.

– Whitehead multiplication of the homotopy groups π_i makes the set $\oplus_i \pi_i$ into a Lie superring. Nobody knows a description of these superrings (what is the semisimple part, what is the radical, etc.). Recently a fantastic description of similar superrings as "positive parts" of certain twisted loops with values in simple **Lie superalgebras** appeared [32].

– Deformations of arbitrary algebraic structures studied by **Gerstenhaber** (as well as the structures themselves) are related with **homological vector fields**, see an ingenious paper by Grusson [24]. Homological fields play also an important role in the formal variational calculus (Gelfand, Dorfman, Daletsky, Tsygan, see [10]).

– The Stinrod algebra in characteristic 2 is identified with the universal enveloping of the "positive part" of the Neveu-Schwarz **Lie superalgebra**, see V. Bukhshtaber's appendix in [8].

– *First order* binary invariant differential operators determine, with a few exceptions, a **Lie superalgebra** structure on their domain [23].

– Quantization of **gauge fields** requires the "odd mechanics" [20].

Selected difficulties and solutions. In [1] **Witten** wrote: Direct experimental confirmation of **supersymmetry** is one of the prime missions of the proposed Superconducting supercollider... More fundamentally, I believe that the main obstacle is that the core geometrical ideas – which must underline *string theory* the way Riemannian geometry underlines *general relativity* – have not yet been unearthed." Actually, by the time Witten wrote this passage a solution – definition of the analog of the curvature tensor for any nonholonomic manifolds or **supermanifolds** – had been unearthed, cf. [21, 22].

Towards noncommutative geometry. To describe physical models, the least one needs is a triple $(X, F(X), L)$, consisting of the "phase space" X, the **sheaf** of functions on it, locally represented by the algebra $F(X)$ of sections of this **sheaf**, and a Lie subalgebra L of the **Lie algebra** of differentiations of $F(X)$ considered as vector fields on X. Here X can be recovered from $F(X)$ as the collection $\mathrm{Spec}(F(X))$, called the spectrum and consisting of maximal, or prime, ideals of $F(X)$. Usually, X is endowed with a suitable topology; for the prime spectra the natural topology (the Zariski one) is very nonHausdorff; since almost any points of such topology is close to any given point this tempts one's imagination to contemplate about possibility of other types of spectra penetrate through walls, etc.

After the discovery of quantum mechanics, the attempts to replace $F(X)$, the algebra of "observables", with a noncommutative ("quantum") algebra A became more and more popular. For a long time, however, nobody, except **Connes**, could develop even elements of differential geometry on spectra of such algebras.

The first successful attempt was *superization* [29, 4] the road to which was prepared in the works of A. Weil, Leray, Grothendiek and Berezin. It turns out that having suitably generalized the notion of the tensor product and differentiation (by inserting certain signs in the conventional formulas) we can reproduce on **supermanifolds** all the **characters** of differential geometry and actually obtain a much reacher and interesting plot than on manifolds. This picture proved to be a great success in theoretical physics since the language of **supermanifolds** and supergroups is a ''natural'' for a uniform description of Bose and fermi particles. Today there is no doubt that this is the language of the Grand Unified Theories of all known fundamental forces, see [11].

Nevertheless, supergroups are not the largest possible symmetries of **superspaces**; there are transformations that preserve more noncommutativity than just a ''mere'' supercommutativity. To be able to observe that there are symmetries that unify Bose and fermi particles we had to admit a broader point of view on our Universe and postulate that we live on a **supermanifold**. In [13] we suggest to consider our **supermanifolds** as particular case of metamanifolds.

How noncommutative should $F(X)$ be? To define the space corresponding to an arbitrary algebra is very hard, see Manin's gloomy remarks in [34], where he studies quadratic algebras as functions on ''perhaps, nonexisting'' noncommutative **projective spaces**.

Manin's idea that there hardly exists one uniform definition suitable for any noncommutative algebra (because there are several quite distinct types of them) was supported by A. Rosenberg's studies. He managed, however, to define several types of spectra in order to interpret ANY algebra as the algebra of functions on a suitable spectrum, see [38]. In particular, there IS a space corresponding to a quadratic (or ''quadraticizable'') algebra such as the so-called ''quantum'' deformation $U_q(\mathfrak{g})$ of $U(\mathfrak{g})$, see [12].

Observe that in [34] Manin also introduced and studied symmetries of *supercommutative superalgebras* **wider than supersymmetries**, but he only considered them in the context of quadratic algebras. Regrettably, nobody, as far as I know, investigated consequences of Manin's approach to enlarging supersymmetries.

Unlike numerous previous attempts to devise noncommutative **algebraic geometry**, Rosenberg's theory appears more natural; still, it is algebraic, without any real geometry (no differential equations, integration, etc.). For some noncommutative algebras certain notions of differential geometry can be generalized: such is, now well-known, A. Connes geometry, see [9], and [35]. Arbitrary algebras seem to be too noncommutative to allow to do any physics.

In contrast, the experience with the simplest *noncommutative spaces*, the superspaces, tells us that all constructions expressible in the language of differential geometry (these are particularly often used in physics) can be carried over to the super case. Still, supersymmetry has certain shortcomings which disappear in the theory of metamanifolds [13].

So, it might seem, that the algebras of functions should be either commutative or supercommutative to allow any interesting geometry. **Drinfeld**'s theory of ''*quantum groups*'' hints that this is not exactly so but to develop the **noncommutative geometry** will not be easy.

Recent results on high temperature superconductivity, presumably carried by **anyons**, particles of fractional spin (whatever that might mean) prepared the audience to these sacreligeous investigations and now, in addition to the beauty of the problems and a natural character of the mathematical formulation I can refer to the authority of physicists, such as **V. Rubakov**.

Selected outstanding results in supermanifold theory and noncommutative geometry: **Serganova**'s description of Kazhdan–Lusztig polynomials and (with **Penkov**) the character formula for a wide class of **irreducible representations** of a wide class of **Lie superalgebras**; **Sergeev**'s invariant theory; **Shander**'s integration theory, nonpolynomial invariant theory and *canonical forms* of various tensors. **Shchepochkina**'s exceptional simple **Lie superalgebras** of vector fields [43] related, perhaps, to the *Standard Model*. On applications of *Kac–Moody superalgebras* or nonpolynomial growth, see **Borcherds** [7] and **Gritsenko** and **Nikulin** [19]. **Egorov**'s *superization* of $\mathfrak{gl}(\infty)$ [15]. Application of noncommutative **algebraic geometry** and theory of $\mathcal{D}$-modules by **Beilinson– Bernstein** and **Brylinsky–Kashivara** to representation theory.

Shortly before his untimely death **Scherk** has demonstrated that supergravity naturally leads to antigravity [42]. Nobody, it seems, investigated this astonishing discovery of a prominet expert.

On notations Good notations help to advance science no less than good theorems. So we should follow best examples (e.g., [11]) and never confuse **Lie superalgebras** with graded **Lie algebras**, *superconformal algebras* with *stringy* superalgebras, never denote the volume element on $4|N$-dimensional supermanifolds ''$d^4xd^N\theta$'' but write vol(x$|\theta$) or $D(x|\theta)$; be aware of numerous formats of *supermatrices*; stick to most meaningful notations of simple **Lie superalgebras** or invent better.

Summary. Supersymmetry naturally leads to supergravity; the arena for the latter is a nonholonomic supermanifold [21]. Quantization naturally leads to noncommutative algebras of functions while the desire to consider the most broad symmetries of the known entities leads to symmetries wider than supersymmetries [13]. Complexification with odd imaginary numbers also leads us to noncommutative algebra of differential geometry. **M. Vasiliev**'s dissident's study of particles with spin >2 lead to a new class of filtered superalgebras, analogs of **Lie algebra** of matrices of *complex size* [14]. Recent discovery by **Shchepochkina** [43] of exceptional **Lie superalgebras** of vector fields gives new insight to the *Standard Models* [46]. Finally, if Duplij's somewhat incomprehencible conjectures in category theory have a germ of reason they might lead to a revolution.

Bibliography

[1] M. Atiyah, A. Borel, G. J. Chaitin, D. Friedan, J. Glimm, J. Gray, M. Hirsch, S. Mac Lane, B. Mandelbrot, D. Ruelle, A. Schwarz, K. Uhlenbeck, R. Thom, E. Witten and C. Zeeman, Responses to: A. Jaffe and F. Quinn, "Theoretical mathematics: toward a cultural synthesis of mathematics and theoretical physics" [Bull. Amer. Math. Soc. (N.S.) 29 (1993), no. 1, 1–13; MR 94h:00007]. Bull. Amer. Math. Soc. (N.S.) 30 (1994), no. 2, 178–20;

[2] F. A. Berezin, The method of second quantization. Translated from the Russian by Nobumichi Mugibayashi and Alan Jeffrey. Pure and Applied Physics, Vol. 24 Academic Press, New York-London 1966 xii+228 pp.; id., The method of second quantization. Second edition. Edited and with a preface by M. K. Polivanov. "Nauka", Moscow, 1986. 320 pp. (Russian);

[3] F. Berezin, Introduction to superanalysis. Edited and with a foreword by A. A. Kirillov. With an appendix by V. I. Ogievetsky. Translated from the Russian by J. Niederle and R. Kotecký. Translation edited by Dimitri Leites. Mathematical Physics and Applied Mathematics, 9. D. Reidel Publishing Co., Dordrecht, 1987. xii+424 pp.;

[4] F. A. Berezin and D. A. Leites, Sov. Math. Doklady, v.16, 1975, 1218–1222.

[5] R. L. Dobrushin et al. (eds.). Contemporary mathematical physics. F. A. Berezin memorial volume. American Mathematical Society Translations, Series 2, 175. Advances in the Mathematical Sciences, 31. American Mathematical Society, Providence, RI, 1996. x+236 pp; id., Topics in statistical and theoretical physics. F. A. Berezin memorial volume. American Mathematical Society Translations, Series 2, 177. Advances in the Mathematical Sciences, 32. American Mathematical Society, Providence, RI, 1996. x+223 pp.

[6] N. Berline, E. Getzler and M. Vergne, Heat kernels and Dirac operators. Grundlehren der Mathematischen Wissenschaften, 298. Springer-Verlag, Berlin, 1992. viii+369 pp.

[7] R. Borcherds, Current developments in mathematics, Cambridge, MA, 1996, 1–36, Int. Press, Boston, MA, 1997; id., Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998). Doc. Math. 1998, Extra Vol. I, 607–615 (electronic); id., Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 744–752, Birkhäuser, Basel, 1995; id., First European Congress of Mathematics, Vol. I (Paris, 1992), 411–421, Progr. Math., 119, Birkhäuser, Basel, 1994; id., Topology 35 (1996), no. 3, 699–710; id., Monstrous moonshine and monstrous Lie superalgebras. Invent. Math. 109 (1992), no. 2, 405–444; Conway, J. H.; Sloane, N. J. A. Sphere packings, lattices and groups. Third edition. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. Grundlehren der Mathematischen Wissenschaften, 290. Springer-Verlag, New York, 1999. xxiv+703 pp.

[8] N. Stinrod and D. Epstein, Cohomology operations (Translation into Russian of Ann. of Math. Studies, 50, Princeton Univ. Press, Princeton, N.J., 1962) With appendices by J. P. May and by V. M. Bukhshtaber, Nauka, Moscow, 1983. 232 pp.

[9] A. Connes, Noncommutative geometry. Academic Press, Inc., San Diego, CA, 1994. xiv+661 pp.

[10] Daletskii, Yu., Funktsional. Anal. i Prilozhen. 20 (1986), no. 2, 62–64. (Russian, English translation: Functional Anal. Appl. 20 (1986), no. 2, 136–138); Daletskii Yu. L., Tsygan B. L., Methods Funct. Anal. Topology 5 (1999), no. 4, 62–86.

[11] P. Deligne, et al (eds.) Quantum fields and strings: a course for mathematicians. Vol. 1, 2. Material from the Special Year on Quantum *Field Theory* held at the Institute for Advanced Study, Princeton, NJ, 1996–1997. AMS, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999. Vol. 1: xxii+723 pp.; Vol. 2: pp. i–xxiv and 727–1501.

[12] V. G. Drinfeld, Proc. Int. Congress Math., Berkeley, **1**, 1986, 798–820.

[13] S. Duplij and J. Wess, Noncommutative Structures in Mathematics and Physics. Proceedings of the NATO Advanced Research Workshop, Kiev, Ukraine September 24–28, 2000, Kluwer, 2001.

[14] L. Brink, T. H. Hansson, S. Konstein and M. A. Vasiliev, Nuclear Phys. B 401 (1993), no. 3, 591–612.

[15] G. Egorov, In: J. Mickelsson, O. Osmo Pekonen (eds) Topological and geometrical methods in *field theory.* Proceedings of the Second International Symposium held in Turku, May 26–June 1, 1991, World Sci. Publishing, River Edge, NJ, 1992, 135–146.

[17] S. I. Gelfand and Yu. I. Manin, Homological algebra. Springer-Verlag, Berlin, 1999. iv+222 pp.

[18] M. Green, J. Schwarz and E. Witten Superstring theory, vv. 1, 2. Second edition. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge-New York, 1988. x+470 pp.; xii+596 pp.

[19] V. Gritsenko and V. Nikulin, Internat. J. Math. 9 (1998), no. 2, 153–199; 201–275.

[20] J. Gomis, J. Paris and S. Samuel, Phys. Rept. 259 (1995) 1–191.

[21] P. Grozman and D. Leites in: J. Wess, E. Ivanov (eds.) Procedings of the Internatnl seminar in the memory of V. Ogievetsky, Dubna 1997, Springer Lect. Notes in Physics, v. 524,1999, 58–67.

[22] P. Grozman and D. Leites, in: E. Ramrez de Arellano, M. Shapiro, L. Tovar and N. Vasilevski (eds.) Proc. Internatnl. Symp. Complex Analysis and related topics, Mexico, 1996, Birkhauser Verlag, 1999, 73–105.

[23] P. Grozman, Funct. Anal. Appl., 1980, v.14, n.2, 58–59; id., JNMP, 2001, v.8, no. 1, 31–37.

[24] C. Gruson, Ann. Inst. Fourier (Grenoble) 50 (2000), no. 3, 807–831.

[25] H. Hertz, The principles of mechanics. Preface by H. von Helmholtz. Translation by D. E. Jones and J. T. Walley. Introduction by Robert S. Cohen. Dover Publications, Inc., New York, 1956. xlii+274 pp.
[26] V. G. Kac, Lie superalgebras. Adv. Math, 1976, 26, 8–96.
[27] F. Klein and Sommerfeld, A., Über die Theorie des Kreisels. (German) Bibliotheca Mathematica Teubneriana, *Band* 1 Johnson Reprint Corp., New York; B. G. Teubner Verlagsgesellschaft, Stuttgart 1965, vi+966 pp.
[28] G. Mandel, In: Kagan V. F. (ed.) Mémoir du Seminair pour l'Analyse vectorielle et tensorielle et pour ses Applications à la Géometries, à la Mécanique et à la Physique, v. 4, ONTI NKTP SSSR, Moscou-Leningrad, 1937, 61–69;
[29] D. Leites, Spectra of graded-commutative rings, Uspehi Matem. Nauk, **29**, no. 3, 1974, 157–158 (in Russian).
[30] D. Leites (ed.), Seminar on supermanifolds, no. 1–34, 2100 pp. Reports of Dept. of Math. of Stockholm Univ., 1986–1990.
[31] A. A. Kirillov, Current problems in mathematics, Vol. 16 (Russian), pp. 3–29, 228, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1980.
[32] C. Löfwall and J.-E. Roos, Adv. Math. 130 (1997), no. 2, 161–200.
[33] Manin Yu., Lectures on algebraic geometry. 1966/67 and 1968/69, Moscow University press, 1968 and 1970. (Russian; an edited version translated into English in [30].)
[34] Manin Yu., *Quantum groups* and noncommutative geometry, CRM, Montreal, 1988.
[35] Manin Yu., Topics in noncommutative geometry, Rice Univ., 1989.
[36] Manin Yu., Gauge field theory and complex geometry. Second edition. Grundlehren der Mathematischen Wissenschaften, 289. Springer-Verlag, Berlin, 1997. xii+346 pp.
[37] V. V. Prasolov, Problems and theorems in linear algebra. Translations of Mathematical Monographs, 134. American Mathematical Society, Providence, RI, 1994. xviii+225 pp.
[38] A. Rosenberg, Noncommutative algebraic geometry and representations of quantized algebras. Mathematics and its Applications, 330. Kluwer, Dordrecht, 1995. xii+315 pp.
[39] D. A. Leites, Funktsional. Anal. i Prilozhen. 14 (1980), no. 2, 35–38; Bernstein, J. N., Leites, D. A., Selecta Math. Soviet. 1 (1981), no. 2, 143–160; V. Serganova, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 583–593.
[40] V. Shander, Funktsional. Anal. i Prilozhen. v. 14, 1980, no. 2, 91–92; Shander, V. N., Funktsional. Anal. i Prilozhen. 17 (1983), no. 1, 89–90.
[41] V. Shander, Invariant functions on *supermatrices*, math.RT/9810112, 24 pp.
[42] J. Scherk, Antigravity: a crazy idea? Phys. Lett. B 88 (1979), no. 3–4, 265–267.
[43] I. M. Shchepochkina, Funktsional. Anal. i Prilozhen. 33 (1999), no. 3, 59–72, 96 (Russian; translation in Funct. Anal. Appl. 33 (1999), no. 3, 208–219 (2000)); id., Represent. Theory 3 (1999), 373–415.
[44] L. Frappat, Dictionary on Lie algebras and superalgebras. Acad. Press., 2000, 432 pp.
[45] A. Konechny and A. Schwarz, hep-th/0012145.
[46] T. A. Larsson, Symmetries of Everything, math-ph/0103013.
[47] V. Molotkov, math.CT/9912060.

Dimitry Leites