Oberseminar p-adische Arithmetik, Wintersemester 2020/21
Moduli of Langlands Parameters
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In this Oberseminar we study the construction and geometric properties of moduli spaces of L-parameters, as described by Dat, Helm, Kurinczuk and Moss in [2]. There are various possible definitions of L-parameters, for the Weil group W_F of a p-adic local field F, with values in the L-group of a reductive group G over F. Over $\bar{\mathbb{Q}}_\ell$ (for a prime $\ell \neq p$) the easiest possibility is to use ℓ-adically continuous L-homomorphisms

$$W_F \longrightarrow {}^L G(\bar{\mathbb{Q}}_\ell).$$

However, with this definition it is less clear how to define a moduli space of such objects. Instead we will use a discretization of the Weil group in order to define a moduli space of L-parameters over $\mathbb{Z}[1/p]$. This moduli space depends on choices used in the discretization, but we will see that the ℓ-adic completions turn out to be independent of these choices and that there exists a universal family of ℓ-adically continuous W_F-representations over these completions. We will study the local and global geometry of these moduli spaces, and the GIT quotient under the action of the dual group, as well as the relation with questions in the context of the local Langlands correspondence.

1) **Various definitions of L-parameters**
The aim of this talk is to recall the four possible definitions of L-parameters given in [2, §§1.1,1.2] (see page 2 for (1),(2),(3) and page 4 for (4)). First treat the story for GL_n: recall the definition of the Weil group of a p-adic local field K (see e.g. [6, (1.1.8)]) and define the Weil-Deligne group and Weil-Deligne representations, [6, §3.1]. State Grothendieck's ℓ-adic monodromy theorem in oder to compare continuous ℓ-adic Galois-representations respectively representations of the Weil group with representations of the Weil-Deligne group.
Next recall some notions and definition concerning reductive groups. In particular the notion of a pinning, the dual group and the definition of the L-group of a reductive group over a (local) field K, see e.g. [1, 11.1]. Define L-homomorphisms and L-parameters (we can for example use the definition as in [3, §4.] but disregard the condition involving *relevant parabolic subgroups* (i.e. we can focus on quasi-split groups); then explain [3, Remark 4.0.2]). Finally explain the equivalence of definitions (1)-(4) of [2].

2) **The general moduli problem, and tame parameters**
Translate L-parameters with values in a $\mathbb{Z}[1/p]$-algebra R into R-valued cocycles as in [2, p.5]. State and prove the representability of the functor $Z^1(W_F^0, \hat{G}(-))$ in general (here we only use the description of L-parameters as in (4) of [2, p.4]. Continuous representations for the ℓ-adic topology and ℓ-adic completion will be treated in talk 3 and 7). Then move to *tame parameters*: cover [2, §§2.1,2.2].

3) **Geometry of the space of tame parameters**
Finish [2, §2]. In particular show that the representing affine scheme of $Z^1(W_F^0/P_F, \hat{G}(-))$ is generically smooth (Prop. 2.7) and ℓ-adically separated (Cor. 2.10). Then construct the universal ℓ-adic W_F/P_F-representation over the ℓ-adic completion of $Z^1(W_F^0/P_F, \hat{G}(-))$, see [2, Theorem 2.12].

4) **Moduli of cocycles, I**
Describe generalities about the moduli spaces of cocycles [2, Appendix A, A.1,A.2,A.3]. Before treating A.3, recall some background material on geometric invariant theory.

5) **Moduli of cocycles, II**
Continue the analysis of moduli schemes of cocycles [2, Appendix A, A.4,A.5,A.6]. Use this abstract machinery to state and prove [2, Theorem 3.1].

6) **Reduction to the tame parameter case**
Cover [2, §3], in particular, prove Theorem 3.4 and Theorem 3.12 of loc. cit.

7) **Geometric properties of the moduli space**
Prove that the moduli scheme $\underline{Z}^1(W_F^0/P_F^e, \hat{G})$ is flat and a local complete intersection; construct the universal ℓ-adic representation over its ℓ-adic completion [2, Theorem 4.1], and prove that the ℓ-adic completion is independent of choices [2, Cor. 4.2]. State Conjectures 4.3 and 4.4 of loc. cit. about connectedness of the moduli spaces. Then cover sections 4.2 and 4.3 of [2]. Prove Theorem 4.13 of loc. cit.

8) **Connected components of the moduli space**
Finish section 4.4 of [2] and cover sections 4.5 and 4.6. In particular, prove Theorem 4.5 and Theorem 4.8.

9) **Deformation theory and unobstructed points**
This talk should cover as much as possible of [2, §§5.1,5.2]. In particular sketch the proof of Theorem 5.5. In order to simplify the exposition we can restrict to the case that $\hat{G}$ has no exceptional factor.

10) **The GIT quotient by the action of $\hat{G}$**
Introduce the notion of *banal primes* [2, §5.3]. Then describe the quotient $\underline{Z}^1(W_F/I_F^e, \hat{G})/\!\!/\hat{G}$, see [2, Thm. 6.7 and Thm. 6.8].

11) **Comparison with Haines' variety of *infinitesimal characters***
Survey Haines description of a a variety structure on the set of infinitesimal characters [3, §§5.1,5.3]. Then identify the GIT quotient of the complex fiber of $\underline{Z}^1(W_F/I_F^e, \hat{G})$ with Haines' variety, see [2, Theorem 6.10].

12) **Comparison with the Bernstein center**
Explain the relation of Haines' variety with the Bernstein center [3, §5.5]. If time permits describe the integral version of this comparison in the case of GL_n. This is the work of Helm and Moss [5], proving [4, Conjecture 7.6].

References

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